# Recent Attempts in the Analysis of Black Hole Radiation

Koichiro Umetsu [1]

*Graduate School of Quantum Science and Technology, Nihon University, Tokyo 101-8308, Japan*

January, 2010

A Dissertation in candidacy for the degree of Doctor of Philosophy

[1] E-mail: umetsu@phys.cst.nihon-u.ac.jp

**Abstract**

In this thesis, we first present a brief review of black hole radiation which is commonly called Hawking radiation. The existence of Hawking radiation by itself is well established by now because the same result is derived by several different methods. On the other hand, there remain several aspects of the effect which have yet to be clarified. We clarify some arguments in previous works on the subject and then attempt to present the more satisfactory derivations of Hawking radiation. To be specific, we examine the analyses in the two recent derivations of Hawking radiation which are based on anomalies and tunneling; both of these derivations were initiated by Wilczek and his collaborators. We then present a simple derivation based on anomalies by emphasizing a systematic use of covariant currents and covariant anomalies combined with boundary conditions which have clear physical meaning. We also extend a variant of the tunneling method proposed by Banerjee and Majhi to a Kerr-Newman black hole by using the technique of the dimensional reduction near the horizon. We directly derive the black body spectrum for a Kerr-Newman black hole on the basis of the tunneling mechanism.

# Contents

# Chapter 1

# Introduction

General theory of relativity and quantum theory are two fundamental theories in modern physics. According to the current understanding of physics, it is well known that all the forces which have been identified in nature can be explained by the electromagnetic force, weak force, strong force and gravity. The first three of them are described by quantum field theory and the remaining gravity is described by the general theory of relativity.

Furthermore, the idea of a unified field theory, which can describe the four forces by a single theory, was advanced. Electromagnetic and weak forces were unified by Weinberg-Salam theory and a grand unified theory which combines the strong interaction with the electroweak interactions was proposed. However, gravity has yet to be successfully included in a theory of everything. A simple attempt to combine the gravitational interaction with the strong and electroweak interactions runs into fundamental difficulties since the resulting theory is not renormalizable. This means that physically meaningful observables contain nonremovable infinities. The string theory has potentiality which solves these problems. However, we have not obtained any solid result in string theory yet and it depends on the future progress. We have not yet formulated a widely accepted, consistent theory that combines the general theory of relativity with the principle of quantum theory. In any case, we must study a framework where both of the general theory of relativity and quantum theory are consistently incorporated.

From the above point of view, the black hole radiation which was suggested by Hawking is very interesting. Although we have not yet confirmed that black holes do really exist, it has been predicted that they exist as a consequence of the general theory of relativity, namely, as special solutions of the basic Einstein equation. According to the Einstein equation, the space-time is curved by the effects of gravity. The space-time curved by a very strong gravity can form a closed

region from which nothing, not even photons, can escape. The closed region is the black hole. Thus black holes cannot classically allow the emission of radiation. However, by using quantum field theory in black hole physics, a mechanism by which black holes can radiate was proposed by Hawking [1, 2]. The radiation from the black hole is commonly called the Hawking radiation. In this sense, it can be said that Hawking radiation is one of precious phenomena where both of the general theory of relativity and quantum theory play a role at the same time. When any new theory of quantum gravity is constructed, it must be checked if it correctly describes Hawking radiation in the proposed theory.

Hawking's original derivation is very direct and physical [2]. The analysis calculates the Bogoliubov coefficients between the in- and out-states for a body collapsing to form a black hole. It is well-known that the characteristic spectrum found in the original derivation agrees with the black body spectrum with a characteristic temperature associated with the black hole if we ignore the back scattering of particles falling into the black hole. Namely, it was found that a black hole behaves as a black body and the black hole emits radiation.

After Hawking's original derivation, various derivations of Hawking radiation have been suggested. All of them reproduce the same result that the black hole entropy is described by the surface area of the black hole and the temperature of the black hole is described by a surface gravity of the black hole. Hawking radiation is thus one of the most striking effects which are widely accepted by now. However, there are several aspects which have not been completely clarified yet. In particular, although the entropy is interpreted as a count of the number of states in statistical mechanics, the entropy of a black hole with a finite temperature has not been derived by counting the number of quantum states associated with the black hole. It is considered that this problem is closely related to the fact that quantum theory of gravity has not been explicitly formulated yet, and it is very difficult to construct a consistent quantum gravity. By examining the various derivations of Hawking radiation, we find that each derivation has both merits and demerits. In this sense, it may be fair to say that these known derivations of Hawking radiation have not reached an impeccable conclusion yet.

Recently, Robinson and Wilczek suggested a new method of deriving Hawking radiation by the consideration of anomalies [3]. The basic idea of the approach is that the flux of Hawking radiation is determined by anomaly cancellation conditions in the background of a Schwarzschild black hole. Iso, Umetsu and Wilczek improved the approach by Robinson and Wilczek, and they extended the method to a charged black hole [4] and a rotating black hole [5]. The approach of Iso, Umetsu

and Wilczek [5] is very transparent and interesting. However, there remain several points to be clarified. We have presented arguments which clarify the basic idea of the derivation and given a simple derivation by using the Ward Identities and boundary conditions [6]. We would like to explain our simple derivation as comprehensibly as possible in the present thesis.

A straightforward derivation on the basis of the tunneling mechanism was also suggested by Parikh and Wilczek [7]. The analysis of tunneling mechanism was mainly confined to the derivation of the temperature of a black hole, and the black body spectrum itself has not been much discussed. More recently, this problem of the black body spectrum was emphasized by Banerjee and Majhi [8]. They showed how to reproduce the black body spectrum directly, which agrees with Hawking's original result, by using the properties of the tunneling mechanism. Thus the derivation on the basis of the tunneling mechanism became more satisfactory. Their result is valid only for black holes with a spherically symmetric geometry. However, it is known that 4-dimensional black holes have not only a mass and a charge but also angular momentum, and the geometry of a rotating black hole becomes spherically asymmetric because of its own rotation. We have recently attempted to extend Banerjee and Majhi's method to a rotating black hole by using a technique valid only near the horizon, which is called the dimensional reduction [9]. We showed that the result agrees with the previous result. We explain our method which shows how to directly derive the black body spectrum for a rotating black hole on the basis of the tunneling mechanism in this thesis. To the best of my knowledge, there is no derivation of the spectrum by using the technique of the dimensional reduction in the tunneling mechanism. Therefore, we believe that this derivation clarifies some aspects of the tunneling mechanism.

The contents of the present thesis are as follows. In Chapter 2, we review some properties of black holes. These properties will be useful to understand the contents of the following chapters. In Chapter 3, we review the original derivation of Hawking radiation by Hawking and briefly explain other representative derivations of Hawking radiation. It will be argued that there are several aspects to be clarified in the existing derivations of Hawking radiation. In Chapter 4, we discussed the derivation of Hawking radiation which is based on anomalies. We clarify some aspects in previous works on this subject and present a simple derivation of Hawking radiation from anomalies. In Chapter 5, we discussed the derivation of Hawking radiation which is based on quantum tunneling. We present a generalization of the derivation of Hawking radiation by Banerjee and Majhi on the basis of the tunneling mechanism to a rotating black hole and also give some clarifying comments. Chapter 6 is devoted to discussion and conclusion. Some of the

technical details are given in appendices.

In this paper, we use the natural system of units

$$c = G = \hbar = 1 \tag{1.0.1}$$

unless stated otherwise, where $c$ is the speed of light in vacuum, $G$ is the gravitational constant and $\hbar$ is the Planck constant (Dirac's constant).

# Chapter 2

# Properties of Black Hole

The existence of black holes has been predicted by the general theory of relativity. To understand Hawking radiation, we have to know some classical properties of black holes first. In this chapter we would like to review some properties of black holes.

The contents of this chapter are as follows. In Section 2.1, we review the general theory of relativity and black holes. We would also like to mention various types of black holes. In Section 2.2, we refer to the Penrose diagram and show how to describe it. In Section 2.3, we discuss how to extract energy from a rotating black hole classically. In Section 2.4, we would like to discuss the dimensional reduction near the event horizon which is a boundary between our universe and a black hole. By using the technique of the dimensional reduction, we show that a 4-dimensional metric associated with a charged and rotating black hole effectively becomes a 2-dimensional spherically symmetric metric. In Section 2.5, we would like to discuss analogies between black hole physics and thermodynamics. In Section 2.6, we review the argument, which was suggested by Bekenstein, that black holes have entropy. These introductory discussions will be useful to understand the contents of the following chapters.

## 2.1 General Theory of Relativity and Black Hole

General theory of relativity is the theory of space-time and gravitation formulated by Einstein in 1915 [10]. The Einstein equation which describes the general theory of relativity is given by [11]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, $g_{\mu\nu}$ is the metric of space-time, $G$ is the gravitational constant, $c$ is the speed of light, and $T_{\mu\nu}$ is the energy-momentum tensor. These

quantities are defined by

$$R \equiv R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\nu\mu} \tag{2.1.2}$$

$$R_{\mu\nu} \equiv R^{\rho}_{\mu\nu\rho} \tag{2.1.3}$$

$$R^{\rho}_{\mu\nu\sigma} \equiv \partial_\nu \Gamma^{\rho}_{\mu\sigma} - \partial_\sigma \Gamma^{\rho}_{\mu\nu} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\rho}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\sigma} \tag{2.1.4}$$

$$\Gamma^{\rho}_{\mu\nu} \equiv \frac{1}{2} g^{\rho\alpha} \left(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\right) \tag{2.1.5}$$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1.6}$$

where $R^{\rho}_{\mu\nu\sigma}$ is the Riemann-Christoffel tensor or the curvature tensor, $\Gamma^{\rho}_{\mu\nu}$ is the Christoffel symbol, and $ds$ is the line element. The expression on the left-hand side of the equation (2.1.1) represents the curvature of space-time as determined by the metric, and the expression on the right-hand side represents the distribution of matter fields. The Einstein equation is then interpreted as a set of equations dictating how the curvature of space-time is related to the distribution of matter and energy in the universe.

It is difficult to solve the general solution for the Einstein equation because the Einstein equation is the quadratic nonlinear differential equation. However, it is known that there are several exact solutions for the Einstein equation. In 1916, Schwarzshild found an exact solution for the Einstein equation [12] which describes the gravitational field outside a black hole which depends only on the mass.

It is considered that black holes are formed as a result of the gravitational collapse of a star with a very large mass. The original stars which will form the black hole have various physical quantities and properties. As soon as a black hole is formed by the gravitational collapse, the state of the black hole becomes a stationary state. It is known that the stationary state is characterized by only three physical parameters, namely, the mass, the angular momentum and the electrical charge. This means that a black hole does not retain the various information of the original star except for these three parameters. In other words, a black hole can uniquely be decided by the mass, the angular momentum and the charge. This consequence is called the black hole uniqueness theorem [13–15] or the no-hair theorem [16], and the uniqueness theorem is shown in a 4-dimensional theory if the solutions of the Einstein equation satisfy the four conditions

1. Only electromagnetic field exits.
2. Asymptotically flat.
3. Stationary.
4. No singularity exists on and outside the event horizon.

Here the fourth condition is based on the cosmic censorship hypothesis proposed by Penrose [17].

Black holes are divided into four groups by depending on parameters and each has its own name (Tab. 2.1). The Schwarzschild black hole depends on only the mass. The Schwarzschild metric, which describes the space-time outside the Schwarzschild black hole, is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \tag{2.1.7}$$

where $r$, $\theta$ and $\varphi$ are commonly used variables in polar coordinates, and $M$ is the mass of the black hole. The Reissner-Nordström black hole depends on both the mass and the charge. The Kerr black hole depends on both the mass and the angular momentum. The Kerr-Newman black hole depends on the mass, the charge and the angular momentum. The Kerr-Newman metric is given by

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta) dt d\varphi \\ &- \frac{a^2 \Delta \sin^2\theta - (r^2 + a^2)^2}{\Sigma} \sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned} \tag{2.1.8}$$

where $a$ is defined in order to adjust the dimensions by

$$a \equiv \frac{L}{M}, \tag{2.1.9}$$

and for simplicity, the symbols are respectively defined by

$$\Sigma \equiv r^2 + a^2 \cos^2\theta, \tag{2.1.10}$$

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2. \tag{2.1.11}$$

In 4 dimensions, the Kerr-Newman black hole is the most general black hole. We can thus obtain the Kerr metric by taking $Q = 0$ in the metric (2.1.8), and we also obtain the Reissner-Nordström metric by taking $L = 0$, namely, $a = 0$. Of course, by taking both $Q = 0$ and $a = 0$ in (2.1.8), it can be checked that the Kerr-Newman metric (2.1.8) actually becomes the Schwarzschild metric (2.1.7). We also note that the metric (2.1.8) is asymptotically flat, i.e., it approaches the Minkowski metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.12}$$

which stands for the flat space-time in our universe.

Tab. 2.1 The types of black holes

| | Non-rotating ($a = 0$) | Rotating ($a \neq 0$) |
|---|---|---|
| Uncharged ($Q = 0$) | Schwarzschild black hole | Kerr black hole |
| Charged ($Q \neq 0$) | Reissner-Nordström black hole | Kerr-Newman black hole |

Here we consider the event horizon which is the surface of the black hole. For the sake of convenience, by using the metric $g_{\mu\nu}$, we describe (2.1.8) as

$$\begin{aligned} ds^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu && (2.1.13) \\ &= g_{tt} dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + 2 g_{t\varphi} dt d\varphi + g_{\varphi\varphi} d\varphi^2, && (2.1.14) \end{aligned}$$

with

$$(g_{\mu\nu}) = \begin{pmatrix} -\dfrac{\Delta - a^2 \sin^2\theta}{\Sigma} & 0 & 0 & -\dfrac{a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta) \\ 0 & \dfrac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\dfrac{a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta) & 0 & 0 & -\dfrac{a^2 \Delta \sin^2\theta - (r^2 + a^2)^2}{\Sigma} \sin^2\theta \end{pmatrix}. \quad (2.1.15)$$

The black hole is the region that even the light cannot escape from its surface. The event horizon of the Kerr-Newman black hole thus appears at the point where $g_{rr} = \infty$, i.e.,

$$\Delta = 0, \qquad \text{(at the horizon)}. \quad (2.1.16)$$

From (2.1.16), the distance from the center of the black hole to the event horizon is given by

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \quad (2.1.17)$$

where we assumed $M^2 > a^2 + Q^2$ since the mass of the black hole is very generally large. By using (2.1.17), $\Delta$ in (2.1.11) can be written as

$$\Delta = (r - r_+)(r - r_-). \quad (2.1.18)$$

There are two event horizons in the case of the Kerr-Newman black hole, i.e., $r_+$ and $r_-$, and they are respectively called the outer event horizon and the inner event horizon. The inner event horizon $r_-$ exists inside the outer event horizon. We do not care the existence of the inner event

horizon since we cannot know information inside the outer horizon. In what follows, we simply describe the outer event horizon as the horizon.

On the horizon $r = r_+$, the metric (2.1.8) becomes the intrinsic metric given by

$$ds^2 = -\frac{a^2\Delta_+ \sin^2\theta - (r_+^2 + a^2)^2}{\Sigma_+} \sin^2\theta d\varphi^2 + \Sigma_+ d\theta^2, \tag{2.1.19}$$

since both $t$ and $r$ are constant on the horizon, i.e., $dt = dr = 0$. Here we defined $\Delta_+ \equiv \Delta(r_+) = 0$ and $\Sigma_+ \equiv \Sigma(r_+)$. We thus find that the area of the black hole $A$ is given by

$$A = \int \sqrt{g_{\theta\theta}(r_+) g_{\varphi\varphi}(r_+)} d\theta d\varphi = 4\pi(r_+^2 + a^2). \tag{2.1.20}$$

From $\Delta_+ = r_+^2 - 2Mr_+ + a^2 + Q^2 = 0$, we can also write the black hole area as

$$A = 4\pi(2Mr_+ - Q^2). \tag{2.1.21}$$

By taking the total differentiation of (2.1.21), we obtain

$$dM = \frac{\kappa}{8\pi} dA + \Omega_{\mathrm{H}} dL + \Phi_{\mathrm{H}} dQ, \tag{2.1.22}$$

where $\kappa$, $\Omega_{\mathrm{H}}$ and $\Phi_{\mathrm{H}}$ are respectively the surface gravity, the angular velocity and the electrical potential on the horizon, which are defined by

$$\kappa \equiv \frac{4\pi(r_+ - M)}{A}, \tag{2.1.23}$$

$$\Omega_{\mathrm{H}} \equiv \frac{4\pi a}{A}, \tag{2.1.24}$$

$$\Phi_{\mathrm{H}} \equiv \frac{4\pi r_+ Q}{A}. \tag{2.1.25}$$

It is known that the relation (2.1.22) is the energy conservation law in black hole physics. It is easy to find that the each term has the dimension of the energy in the natural system of units.

Before closing this section, we would like to state black holes in various dimensions. It is known that a vacuum solution of the Einstein equation without the cosmological constant in three dimensions ((2+1)-dimensions), corresponds to a flat solution and no black hole solution exists. However, when we consider the Einstein equation with a negative cosmological constant which behaves as attraction, we can obtain black hole solutions in 3-dimensions. It is called the BTZ black hole, which was found by Bañados, Teitelboim and Zanelli [19, 57]. It is known that it is the lowest dimensional black hole.

In dimensions higher than four, there are several black hole solutions because the restriction of topology with respect to the horizon is alleviated. Therefore, a black hole cannot be uniquely

decided even if the mass, the angular momentum and the charge are given. This suggests that the uniqueness theorem is not satisfied. For example, in five dimensions, there are the Myers-Perry black hole which has two independent rotation parameters [20] and the black ring [21]. We thus have these two solutions with the same mass and the same angular momenta. It is thus known that the black hole uniqueness theorem does not hold in the higher dimensional theory.

## 2.2 Penrose Diagram

Penrose diagram is very useful to understand the global structure of black hole space-time. It was proposed by Penrose in 1964 [22]. In this section, we would like to recall the advantages of using the Penrose diagram. Then we will show how to describe the Penrose diagram.

For simplicity, we consider the case of the Schwarzschild black hole. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2, \tag{2.2.1}$$

where $d\Omega^2$ stands for a 2-dimensional unit sphere defined by

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2. \tag{2.2.2}$$

It follows from the expression (2.2.1) that there are two singularities $r = 0$ and $r = 2M$ in the Schwarzschild metric. A singularity at $r = 0$ is the curvature singularity which cannot be removed while the other at $r = 2M$ is a fictitious singularity arising merely from an improper choice of coordinates. We therefore know that the singularity at $r = 2M$ can be removed by using appropriate coordinates.

The Penrose diagram is drawn for the Schwarzschild metric as in Fig. 2.1. The notations $\mathcal{I}^0$, $\mathcal{I}^\pm$ and $\mathcal{J}^\pm$ appearing in Fig. 2.1, respectively stand for the following regions

$$\mathcal{I}^0 = \begin{cases} t\ ;\ \text{finite} \\ r \to \infty\ , \end{cases} \qquad \mathcal{I}^\pm = \begin{cases} t \to \pm\infty \\ r\ ;\ \text{finite}\ , \end{cases} \tag{2.2.3}$$

$$\mathcal{J}^- = \begin{cases} t \to -\infty \\ r \to +\infty\ , \end{cases} \qquad \mathcal{J}^+ = \begin{cases} t \to +\infty \\ r \to +\infty\ , \end{cases} \tag{2.2.4}$$

and two double lines of $\mathcal{R}$ stand for the curvature singularity of the Schwarzschild metric. The heavy lines $\mathcal{H}^+$ and $\mathcal{H}^-$ also stand for

$$\mathcal{H}^+ = \begin{cases} t \to +\infty \\ r = 2M\ , \end{cases} \qquad \mathcal{H}^- = \begin{cases} t \to -\infty \\ r = 2M\ , \end{cases} \tag{2.2.5}$$

and $\mathcal{H}^+$ and $\mathcal{H}^-$ are respectively called the future event horizon and the past event horizon.

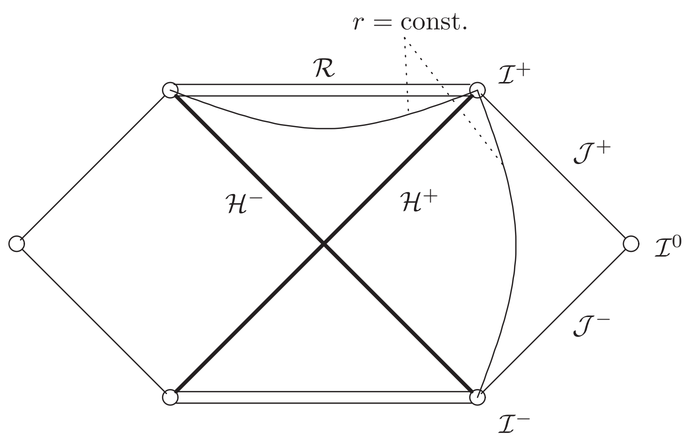


Fig. 2.1 The Penrose diagram for the Schwarzschild solution.

We can draw the Penrose diagram through some coordinate transformations (see, for example, [23]). As a first step of coordinate transformations, we use the tortoise coordinate defined by [24,25]

$$dr_* \equiv \frac{1}{1-\frac{2M}{r}}dr. \tag{2.2.6}$$

The metric (2.2.1) is then written by

$$ds^2 = -\left(1-\frac{2M}{r}\right)(dt-dr_*)(dt+dr_*)+r^2d\Omega^2. \tag{2.2.7}$$

By integrating (2.2.6) over $r$ from 0 to $r$, we obtain

$$r_* = r+2M\ln\left|\frac{r}{2M}-1\right|. \tag{2.2.8}$$

As the second step we use the Eddington-Finkelstein coordinates defined by [26, 27]

$$\begin{cases} v \equiv t+r_* = t+r+2M\ln\left|\frac{r}{2M}-1\right|, \\ u \equiv t-r_* = t-r-2M\ln\left|\frac{r}{2M}-1\right|, \end{cases} \tag{2.2.9}$$

where $v$ is called the advanced time and $u$ is called the retarded time. The metric (2.2.7) is then written as

$$ds^2 = -\left(1-\frac{2M}{r}\right)dvdu+r^2d\Omega^2. \tag{2.2.10}$$

As the third step we use the Kruskal-Szekeres coordinates [28, 29]. When $r > 2M$, these coordinates are defined by

$$\begin{cases} V \equiv \exp\left[\dfrac{v}{4M}\right], \\ U \equiv -\exp\left[-\dfrac{u}{4M}\right], \end{cases} \tag{2.2.11}$$

and the metric (2.2.10) is written as

$$ds^2 = -\frac{32M^3}{r}\exp\left[-\frac{r}{2M}\right] dV dU + r^2 d\Omega^2, \qquad \text{when} \quad r > 2M. \tag{2.2.12}$$

When $r < 2M$, these coordinates are defined by

$$\begin{cases} V \equiv \exp\left[\dfrac{v}{4M}\right], \\ U \equiv \exp\left[-\dfrac{u}{4M}\right], \end{cases} \tag{2.2.13}$$

and the metric (2.2.10) is then written as

$$ds^2 = \frac{32M^3}{r}\exp\left[-\frac{r}{2M}\right] dV dU + r^2 d\Omega^2, \qquad \text{when} \quad r < 2M. \tag{2.2.14}$$

As the fourth step we use the following coordinate transformations defined by

$$\begin{cases} \tilde{V} = \tan^{-1}\left(\dfrac{V}{4M\sqrt{2M}}\right), \\ \tilde{U} = \tan^{-1}\left(\dfrac{U}{4M\sqrt{2M}}\right). \end{cases} \tag{2.2.15}$$

We find that infinities appeared in $V$ or $U$ are converted to finite values such as $\dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$.

As the final step we use the following coordinate transformations defined by

$$\begin{cases} \tilde{T} = \dfrac{1}{2}\left(\tilde{V} + \tilde{U}\right), \\ \tilde{R} = \dfrac{1}{2}\left(\tilde{V} - \tilde{U}\right). \end{cases} \tag{2.2.16}$$

Penrose diagram is drawn by choosing the vertical axis as $\tilde{T}$ and the horizontal axis as $\tilde{R}$.

As an illustration, we draw $\mathcal{I}^+$ and $\mathcal{J}^+$. First, $\mathcal{I}^+$ is expressed by

$$\mathcal{I}^+ = \begin{cases} t \to +\infty \\ r \ ; \ \text{finite} \ . \end{cases} \tag{2.2.17}$$

Since $r$ is finite, we need to consider two cases of $r > 2M$ and $r < 2M$. When $r > 2M$, by substituting (2.2.17) into (2.2.9), $v$ and $u$ become

$$\mathcal{I}^+ = \begin{cases} v \to +\infty \\ u \to +\infty \ . \end{cases} \tag{2.2.18}$$

While when $r < 2M$, by substituting (2.2.17) into (2.2.13), $v$ and $u$ agree with (2.2.18). By substituting (2.2.18) into (2.2.11), $V$ and $U$ become

$$\mathcal{I}^+ = \begin{cases} V \to +\infty \\ U \to 0 \end{cases}, \tag{2.2.19}$$

and by substituting (2.2.19) into (2.2.15), $\tilde{V}$ and $\tilde{U}$ become

$$\mathcal{I}^+ = \begin{cases} \tilde{V} \to +\dfrac{\pi}{2} \\ \tilde{U} \to 0 \end{cases}. \tag{2.2.20}$$

By substituting (2.2.20) into (2.2.16), $\tilde{T}$ and $\tilde{R}$ become

$$\mathcal{I}^+ = \begin{cases} \tilde{T} \to +\dfrac{\pi}{4} \\ \tilde{R} \to +\dfrac{\pi}{4}. \end{cases} \tag{2.2.21}$$

We thus find that the region $\mathcal{I}^+$ as in (2.2.17) is represented by $(\tilde{R}, \tilde{T}) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ in the Penrose diagram, when $r$ takes finite values except $r = 2M$ (Fig. 2.2).

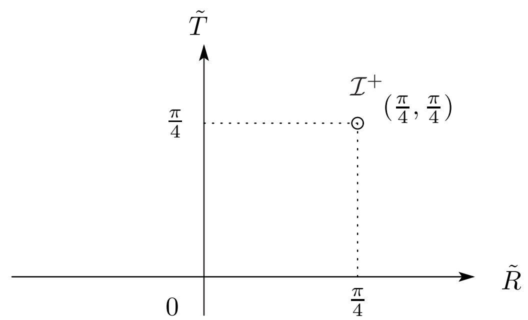


Fig. 2.2 The region of $\mathcal{I}^+$ in the Penrose diagram.

Next, we similarly draw $\mathcal{J}^+$. The region $\mathcal{J}^+$ is expressed by

$$\mathcal{J}^+ = \begin{cases} t \to +\infty \\ r \to +\infty \\ u\ ;\ \text{finite}\ . \end{cases} \tag{2.2.22}$$

Since $r$ is at infinity, we have only to consider the case of $r > 2M$. By substituting (2.2.22) into (2.2.9), $v$ and $u$ become

$$\mathcal{J}^+ = \begin{cases} v \to +\infty \\ u\ ;\ \text{finite}\ . \end{cases} \tag{2.2.23}$$

By substituting (2.2.23) into (2.2.11), $V$ and $U$ become

$$\mathcal{J}^+ = \begin{cases} V \to +\infty \\ U \ ; \ \text{finite} \ , \end{cases} \tag{2.2.24}$$

and by substituting (2.2.24) into (2.2.15), $\tilde{V}$ and $\tilde{U}$ become

$$\mathcal{J}^+ = \begin{cases} \tilde{V} \to +\dfrac{\pi}{2} \\ \tilde{U} \ ; \ \text{finite} \ . \end{cases} \tag{2.2.25}$$

Finally, by substituting (2.2.25) into (2.2.16), $\tilde{T}$ and $\tilde{R}$ become

$$\mathcal{J}^+ = \begin{cases} \tilde{T} = \dfrac{1}{2}\left(\dfrac{\pi}{2} + \tilde{U}\right) \\ \tilde{R} = \dfrac{1}{2}\left(\dfrac{\pi}{2} - \tilde{U}\right) . \end{cases} \tag{2.2.26}$$

From these two relations (2.2.26), we thus find that the region $\mathcal{J}^+$ as in (2.2.22) is represented by the segment of a line

$$\tilde{T} = \frac{\pi}{2} - \tilde{R} \tag{2.2.27}$$

in the Penrose diagram (Fig. 2.3).

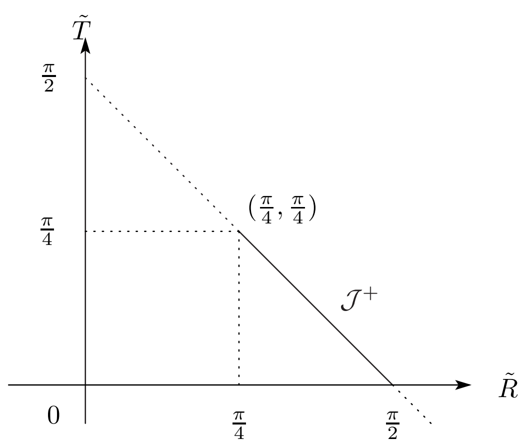


Fig. 2.3 The region of $\mathcal{J}^+$ in the Penrose diagram.

We can similarly draw other points and segments (Tab. 2.2). In Tab. 2.2, when a variable is finite and is not uniquely fixed, the name of the variable is retained. The diagram drawn by using

Tab. 2.2, is expressed as in Fig. 2.4. The regions $\mathcal{R}^+$ and $\mathcal{R}^-$ respectively stand for the following regions

$$\mathcal{R}^+ = \begin{cases} t \to +\infty \\ r = 0 \end{cases}, \qquad \mathcal{R}^- = \begin{cases} t \to -\infty \\ r = 0 \end{cases}, \tag{2.2.28}$$

and the double line $\mathcal{R}$ combines between $\mathcal{R}^+$ and $\mathcal{R}^-$. The region $\mathcal{R}$ stands for $r = 0$ with finite $t$, but we cannot uniquely decide the point in the region $\mathcal{R}$. This means that we do not know how to draw an exact line of the region $\mathcal{R}$. We therefore drew a double line as the line $\mathcal{R}$. Also by comparison with Fig. 2.1, there are some missing parts in Fig. 2.4. We can draw them by defining the other universe where time proceeds reversely by comparison with our universe. We however skip them because they are not important in the body of the present thesis.

Tab. 2.2 Coordinate values in each region

| Region | $(t, r)$ | $(v, u)$ | $(V, U)$ | $\left(\tilde{V}, \tilde{U}\right)$ | $\left(\tilde{T}, \tilde{R}\right)$ |
|---|---|---|---|---|---|
| $\mathcal{I}^+$ | $(+\infty, r)$ | $(+\infty, +\infty)$ | $(+\infty, 0)$ | $\left(+\frac{\pi}{2}, 0\right)$ | $\left(+\frac{\pi}{4}, +\frac{\pi}{4}\right)$ |
| $\mathcal{I}^-$ | $(-\infty, r)$ | $(-\infty, -\infty)$ | $(0, -\infty)$ | $\left(0, -\frac{\pi}{2}\right)$ | $\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)$ |
| $\mathcal{I}^0$ | $(t, +\infty)$ | $(+\infty, -\infty)$ | $(+\infty, -\infty)$ | $\left(+\frac{\pi}{2}, -\frac{\pi}{2}\right)$ | $\left(0, +\frac{\pi}{2}\right)$ |
| $\mathcal{J}^+$ | $(+\infty, +\infty)$ | $(+\infty, u)$ | $(+\infty, U)$ | $\left(\frac{\pi}{2}, \tilde{U}\right)$ | $\tilde{T} = \frac{\pi}{2} - \tilde{R}$ |
| $\mathcal{J}^-$ | $(-\infty, +\infty)$ | $(v, -\infty)$ | $(V, -\infty)$ | $\left(\tilde{V}, -\frac{\pi}{2}\right)$ | $\tilde{T} = \tilde{R} - \frac{\pi}{2}$ |
| $\mathcal{H}^+$ | $(+\infty, 2M)$ | $(v, +\infty)$ | $(V, 0)$ | $\left(\tilde{V}, 0\right)$ | $\tilde{T} = \tilde{R}$ |
| $\mathcal{H}^-$ | $(-\infty, 2M)$ | $(-\infty, u)$ | $(0, U)$ | $\left(0, \tilde{U}\right)$ | $\tilde{T} = -\tilde{R}$ |
| $\mathcal{R}^+$ | $(+\infty, 0)$ | $(+\infty, +\infty)$ | $(+\infty, 0)$ | $\left(+\frac{\pi}{2}, 0\right)$ | $\left(+\frac{\pi}{4}, +\frac{\pi}{4}\right)$ |
| $\mathcal{R}^-$ | $(-\infty, 0)$ | $(-\infty, -\infty)$ | $(0, \infty)$ | $\left(0, +\frac{\pi}{2}\right)$ | $\left(+\frac{\pi}{4}, -\frac{\pi}{4}\right)$ |

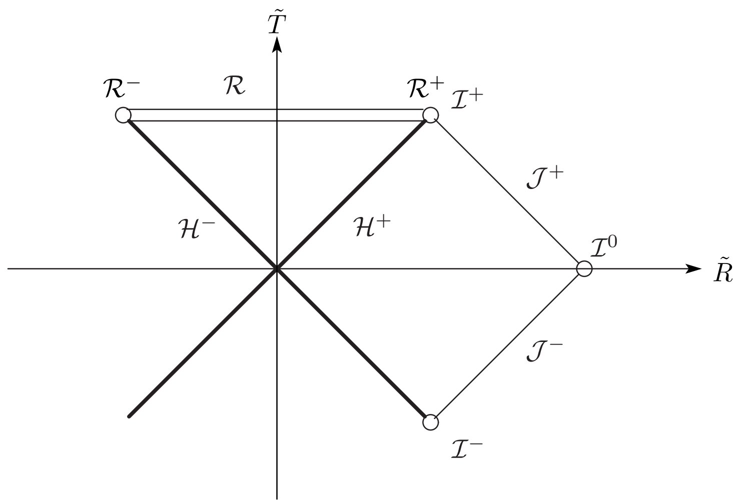


Fig. 2.4 The Penrose diagram corresponding to Tab. 2.1.

As above, the Penrose diagram can represent infinite time or radial coordinates as points or lines. In this diagram null geodesics is also represented as lines of $\pm 45^\circ$ to the vertical. Each point of the diagram represents a 2-dimensional sphere of area $4\pi r^2$. Namely, angular coordinates $\theta$ and $\varphi$ as in (2.2.1) are attached to each point of the coordinate. For this reason, the Penrose diagram is also called the conformal diagram.

The Penrose diagram is divided into four regions by the two diagonal lines $\mathcal{H}^+$ and $\mathcal{H}^-$ (Fig. 2.5). The region I represents our universe. The region Ⅱ represents a black hole. The region Ⅲ represents the other universe that time reversely proceeds by comparison with our universe. The region Ⅳ represents a white hole which is the time reversal of a black hole and ejects matter from the horizon. For example, we find that null geodesics in the region I can arrive at $\mathcal{J}^+$ or the black hole through the horizon $\mathcal{H}^+$ but null geodesics in the region Ⅱ (inside the black hole) cannot arrive at our universe through the horizon $\mathcal{H}^+$.

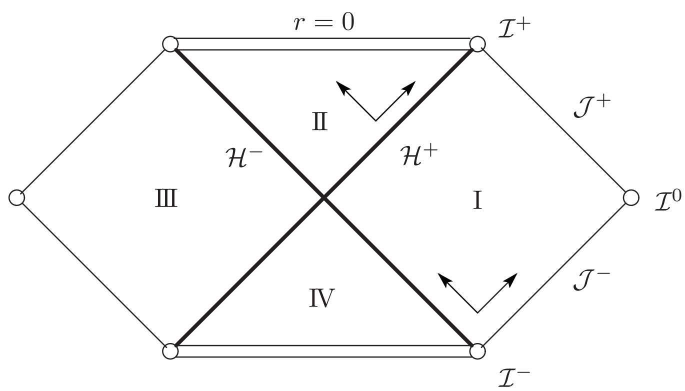


Fig. 2.5 The Penrose diagram for the Schwarzschild solution.

Now we consider that a black hole is formed by the gravitational collapse of a star with a heavy mass. This was comprehensibly discussed by Hawking in the literature [2]. Hence we would faithfully like to present the argument by following Hawking's exposition. For simplicity, we assume that the gravitational collapse is spherically symmetric. Such a object starts to collapse at the point $\mathcal{I}^-$. Since the collapsing object has a mass, the passing is later than light (Fig. 2.6). In Fig. 2.6, the time-like geodesic with an angle, which is smaller than $45°$, represents the surface of the collapsing object and the shaded region represents inside the collapsing object.

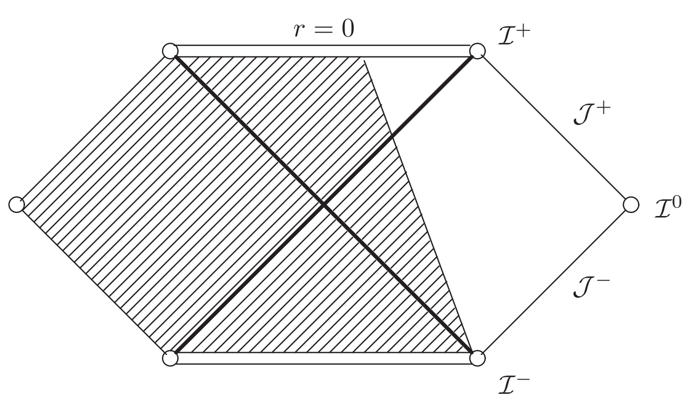


Fig. 2.6 The development of the collapsing object in the Penrose diagram.

In the case of exactly spherical collapse, the metric is exactly the Schwarzschild metric everywhere outside the surface of the collapsing object. O the other hand, inside the object the metric

is completely different. Thus, the past event horizon, the past curvature singularity and the other asymptotically flat region do not exist and are replaced by a time-like curve representing the origin of polar coordinates. The appropriate Penrose diagram is shown in Fig. 2.7. We represented the origin as the vertical dotted line because the metric inside the object might be nonsingular at the origin.

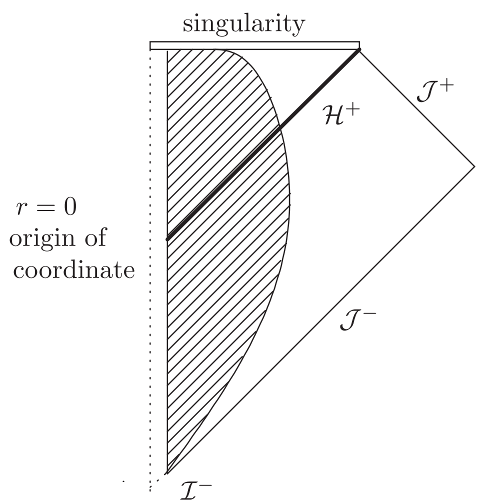


Fig. 2.7 The Penrose diagram of a spherically symmetric collapsing body producing a black hole.

## 2.3 Energy Extraction from Rotating Black Holes

By definition, a black hole is a "region of no escape." It might thus seem that energy cannot be extracted from a black hole. However, the mechanism of energy extraction from a rotating black hole was proposed by Penrose [30]. The process is called the Penrose process and the radiance is called the black hole superradiance. This can be explained in the classical theory.

### 2.3.1 Penrose process

In this subsection, we would like to show how to extract energy from a rotating black hole. To begin with, we shall present an intuitive explanation. The metric of a rotating black hole is given

by the Kerr metric

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta}{\Sigma}(r^2+a^2-\Delta)dtd\varphi \\ &-\frac{a^2\Delta\sin^2\theta-(r^2+a^2)^2}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2. \end{aligned} \tag{2.3.1}$$

This form agrees with the Kerr-Newman metric (2.1.8) but the contents of both $\Delta$ and $\Sigma$ are different because of $Q = 0$. The event horizon in the Kerr coordinate system is defined by $g_{rr} = \infty$ except at the curvature singularity. The surface defined by $g_{tt} = 0$ is called the *ergosphere*. The region enclosed by the ergosphere and the event horizon is called the ergoregion (Fig. 2.8). In the case of the Schwarzschild metric, the ergosphere agrees with the event horizon.

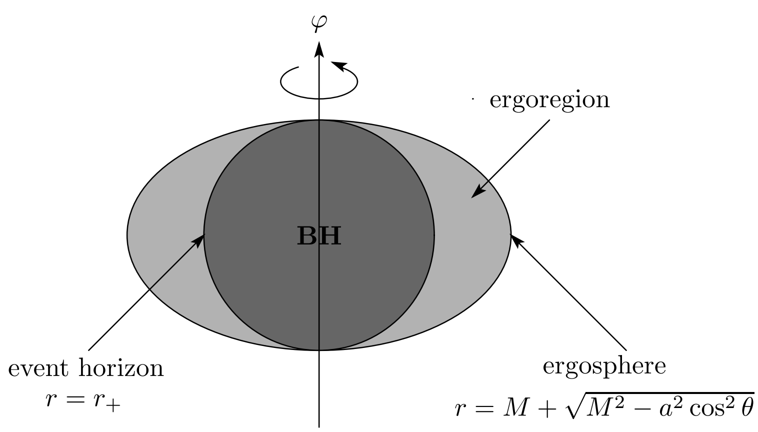


Fig. 2.8 Event horizon and ergosphere

We consider a process where a particle breaks up into two fragments in the ergoregion. The energy of the original particle at infinity is represented as $E_0$. By defining the four dimensional momentum of the particle as $p_0^\mu$, the energy is given by

$$E_0 = -p_0^\mu \xi_\mu, \tag{2.3.2}$$

where $\xi^\mu$ is the Killing field defined by

$$\xi^\mu \equiv \left(\frac{\partial}{\partial t}\right)^\mu, \tag{2.3.3}$$

which becomes a time translation asymptotically at infinity and is space-like in the ergoregion. When the particle enters the ergoregion, we arrange to have it break up into two fragments (Fig. 2.9). By the local momentum conservation law, we have

$$p_0^\mu = p_1^\mu + p_2^\mu, \tag{2.3.4}$$

where $p_1^\mu$ and $p_2^\mu$ are the four dimensional momenta of the two fragments. By contracting the equation (2.3.4) with $\xi_\mu$, we obtain the local energy conservation law

$$E_0 = E_1 + E_2. \tag{2.3.5}$$

The energy need not be positive in the ergoregion since $\xi^\mu$ is space-like there. We can arrange the breakup so that one of the fragments has negative total energy,

$$E_1 < 0. \tag{2.3.6}$$

The fragment with the negative energy falls into the black hole through the event horizon, while the other can escape to infinity since it does not pass through the event horizon. Therefore, we can obtain

$$E_2 > E_0. \tag{2.3.7}$$

This means that energy can be classically extracted from a black hole. The above process is called the Penrose process.

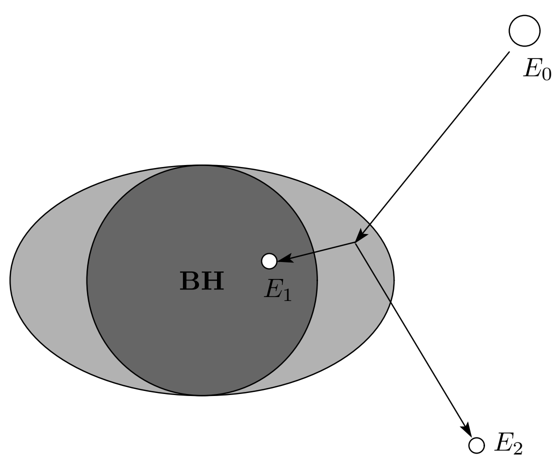


Fig. 2.9 Energy extraction from black hole.

Of course, all energy cannot be extracted from the black hole by the Penrose process. The negative energy particle also carry a negative angular momentum, i.e., the angular momentum opposite to that of the black hole. As a result, the black hole gradually decreases its angular momentum. When the black hole loses the total angular momentum, it becomes a Schwarzschild black hole. Since the ergosphere no longer exist in the case of a Schwarzschild black hole, no further energy extraction can occur.

To see the limit on energy extraction, we use the Killing field $\chi^\mu$ defined by

$$\chi^\mu \equiv \xi^\mu + \Omega_{\rm H}\psi^\mu, \tag{2.3.8}$$

where $\Omega_{\rm H}$ is the angular velocity defined by (2.1.24) and $\psi^\mu$ is the axial Killing field defined by

$$\psi^\mu \equiv \left(\frac{\partial}{\partial\varphi}\right)^\mu. \tag{2.3.9}$$

The Killing field is tangent to the null geodesic generators of the horizon and is future directed null on the horizon. Since the Killing field (2.3.8) is future directed null on the horizon and $p^\mu$ is future-directed timelike or null, we have

$$-p^\mu\chi_\mu \geq 0. \tag{2.3.10}$$

By substituting (2.3.8) into (2.3.10), we obtain

$$-p^\mu(\xi_\mu + \Omega_{\rm H}\psi_\mu) = \omega - m\Omega_{\rm H} \geq 0, \tag{2.3.11}$$

where $\omega$ is the energy of the fragment which enters the black hole and $m = p^\mu\psi_\mu$ is an angular momentum of it. In a Kerr black hole background, the system is stationary and has the axial symmetry. We therefore find that both the energy and the angular momentum are conserved and these quantities are identified at asymptotic infinity (the Minkowski space). The relation (2.3.11) is also written as

$$m \leq \frac{\omega}{\Omega_{\rm H}}. \tag{2.3.12}$$

If $\omega$ is negative, $m$ is also negative. Thus the angular momentum of the black hole is reduced. The mass and the angular momentum of the black hole are respectively $M + \delta M$ and $L + \delta L$ where $\delta M = \omega$ and $\delta L = m$. Thus we obtain

$$\delta L \leq \frac{\delta M}{\Omega_{\rm H}} = \frac{2M\left(M^2 + \sqrt{M^4 - L^2}\right)}{L}\delta M, \tag{2.3.13}$$

where we used the formula for $\Omega_{\rm H}$. This is equivalent to

$$\delta\left(\frac{1}{2}\left[M^2+\sqrt{M^4-J^2}\right]\right)\geq 0. \tag{2.3.14}$$

Christodoulou defined the irreducible mass $M_{\rm ir}$ by [31]

$$M_{\rm ir}^2 \equiv \frac{1}{2}\left[M^2+\sqrt{M^4-J^2}\right] \tag{2.3.15}$$

$$= \frac{1}{2}\left[M^2+M\sqrt{M^2-a^2}\right]. \tag{2.3.16}$$

The irreducible mass can also be written in terms of the black hole area as in (2.1.20), i.e.,

$$M_{\rm ir}^2 = \frac{A}{16\pi}. \tag{2.3.17}$$

By substituting (2.3.17) into (2.3.14), the energy extraction by Penrose process is thus limited by the requirement that

$$\delta A \geq 0. \tag{2.3.18}$$

This result agrees with Hawking's black hole area theorem that the black hole area never decreases [32].

### 2.3.2 Superradiance

There is a wave analog of the Penrose process [33, 34]. It is called superradiant scattering or superradiance. It is known that scalar fields display superradiance. To find this, we consider the energy current defined by

$$J_\mu \equiv -T_{\mu\nu}\xi^\nu, \tag{2.3.19}$$

where $T_{\mu\nu}$ is an energy-momentum tensor which is a symmetric tensor in this case. We take the covariant derivative $\nabla^\mu$ of (2.3.19)

$$\nabla^\mu J_\mu = -(\nabla^\mu T_{\mu\nu})\xi^\nu - T_{\mu\nu}(\nabla^\mu\xi^\nu). \tag{2.3.20}$$

By the general coordinate invariance, the energy-momentum tensor satisfies

$$\nabla^\mu T_{\mu\nu} = 0. \tag{2.3.21}$$

We thus find that the relation (2.3.20) becomes

$$\nabla^\mu J_\mu = -T_{\mu\nu}(\nabla^\mu\xi^\nu) \tag{2.3.22}$$

$$= -\frac{1}{2}T_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) \tag{2.3.23}$$

$$= 0, \tag{2.3.24}$$

where we used the facts that the energy-momentum tensor is a symmetric tensor and Killing fields satisfy the Killing equation

$$\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu = 0. \tag{2.3.25}$$

If we integrate (2.3.24) over the region $\mathcal{K}$ of space-time whose boundary consists of two spacelike hypersurfaces $\Sigma_1$ at $t$ and $\Sigma_2$ at $t + \delta t$ (the constant time slice $\Sigma_2$ is a time translate of $\Sigma_1$ by $\delta t$) and two timelike hypersurfaces $\mathcal{H}$ (the event horizon at $r = r_+$) and $\mathscr{S}(\infty)$ (large sphere at spatial infinity $r \to \infty$), we can know the presence or absence of the superradiance. The intuitive figure is shown in Fig. 2.10. Strictly speaking, this figure is not precise. The precise figure is shown in Fig. 2.11 by using Penrose diagram.

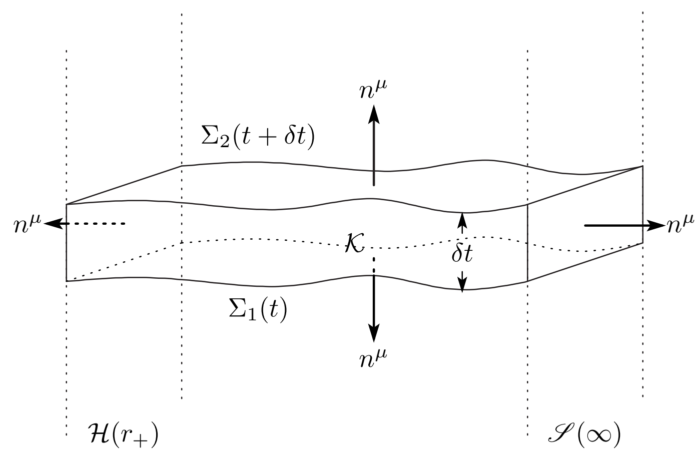


Fig. 2.10 Intuitive figure with respect to Gauss's theorem

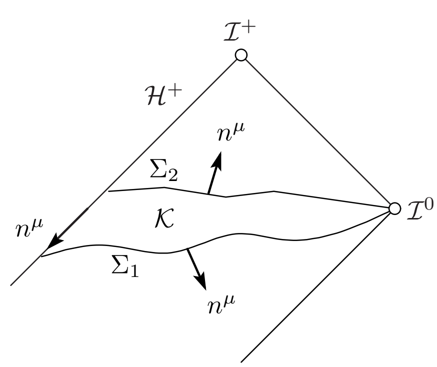


Fig. 2.11 Precise figure using the Penrose diagram

By using Gauss's theorem, we obtain

$$0 = \int_{\mathcal{K}} \sqrt{-g} d^4x (\nabla_\mu J^\mu) \tag{2.3.26}$$

$$= \int_{\partial\mathcal{K}} d\Sigma_\mu J^\mu \tag{2.3.27}$$

$$= \int_{\Sigma_1(t)} n_\mu J^\mu d\Sigma + \int_{\Sigma_2(t+\delta t)} n_\mu J^\mu d\Sigma + \int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma + \int_{\mathscr{S}(\infty)} n_\mu J^\mu d\Sigma, \tag{2.3.28}$$

where $\partial\mathcal{K}$ is the boundary of the region $\mathcal{K}$, $d\Sigma_\mu \equiv n_\mu d\Sigma$ is a 3-dimensional suitable area element and the unit vector $n^\mu$ is outwardly normal to the region $\mathcal{K}$. In the last line, the first two terms cancel with each other because the system has the time translation symmetry and the two directions of $n_\mu$ are opposite to each other. The third term represents the flow of the net energy current flux into the black hole. The last term represents the net energy current flux flow out of $\mathcal{K}$ to infinity. Thus the relation (2.3.28) becomes

$$\int_{\mathscr{S}(\infty)} n_\mu J^\mu d\Sigma = -\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma. \tag{2.3.29}$$

If the quantity on the right-hand side in (2.3.29) is positive (negative), this means that the outgoing energy current flux is larger (smaller) than the incident one and the superradiance is present (absent).

We would like to evaluate the quantity on the right-hand side in (2.3.29). The vector $n^\mu$ is normal to the event horizon. The normal vector $n^\mu$ can be written in terms of the Killing field $\chi^\mu$ as

$$n^\mu = -\chi^\mu, \tag{2.3.30}$$

where $\chi^\mu$ is the Killing field defined by (2.3.8). As already stated, one may recall that the Killing field is tangent to the horizon. One might therefore wonder the appearance of the relation (2.3.30). This result is known by the fact that the vector which is normal to the horizon is tangent to itself on the horizon (the null hypersurface). We show a proof in Appendix A. The sign of (2.3.30) is decided by the direction of $n^\mu$ toward the horizon which is opposite to the future directed Killing field. We thus obtain

$$\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma = -\int_{\mathcal{H}(r_+)} \chi_\mu J^\mu d\Sigma \tag{2.3.31}$$

$$= -\int_{\mathcal{H}(r_+)} \chi_\mu \left(-T^\mu_{\ \nu}\xi^\nu\right) d\Sigma \tag{2.3.32}$$

$$= \int_{\mathcal{H}(r_+)} \chi^\mu T_{\mu\nu}\xi^\nu d\Sigma, \tag{2.3.33}$$

where we used the definition (2.3.19).

Here we would like to find a concrete form of the energy-momentum tensor $T_{\mu\nu}$. For sake of simplicity, we consider the action for a massless scalar field without interactions. In curved space-time, the action is given by

$$S \equiv \int \sqrt{-g} d^4x \, [\mathcal{L}] \tag{2.3.34}$$

$$= \int \sqrt{-g} d^4x \left[ \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right], \tag{2.3.35}$$

where $\mathcal{L}$ is the Lagrangian density. According to field theory, the energy-momentum tensor $T_{\mu\nu}$ is then defined by

$$T_{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial \left( \nabla^\mu \phi \right)} \nabla_\nu \phi - g_{\mu\nu} \mathcal{L} \tag{2.3.36}$$

$$= \frac{1}{2} (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \phi)(\nabla^\alpha \phi), \tag{2.3.37}$$

where we used the Lagrangian density in (2.3.35). By substituting (2.3.37) into (2.3.33), we obtain

$$\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma = \int_{\mathcal{H}(r_+)} d\Sigma \left[ \frac{1}{2} \left( \chi^\mu \nabla_\mu \phi \right) \left( \xi^\mu \nabla_\mu \phi \right) - \frac{1}{2} \chi^\mu \xi_\mu \left( \nabla_\alpha \phi \right) \left( \nabla^\alpha \phi \right) \right] \tag{2.3.38}$$

$$= \int_{\mathcal{H}(r_+)} d\Sigma \left[ \frac{1}{2} \left( \chi^\mu \nabla_\mu \phi \right) \left( \xi^\mu \nabla_\mu \phi \right) \right], \tag{2.3.39}$$

where we used the fact that $\chi^\mu \xi_\mu = 0$ on the horizon. Since we consider the case of a Kerr black hole which is stationary and axisymmetric, the scalar field can be written asymptotically as

$$\phi(x) = \phi_0(r, \theta) \cos(\omega t - m\varphi). \tag{2.3.40}$$

Also we asymptotically have

$$\chi^\mu \nabla_\mu = \frac{\partial}{\partial t} + \Omega_{\rm H} \frac{\partial}{\partial \varphi}, \tag{2.3.41}$$

$$\xi^\mu \nabla_\mu = \frac{\partial}{\partial t}. \tag{2.3.42}$$

We then find that the integrand of (2.3.39) asymptotically becomes

$$\frac{1}{2} \left( \chi^\mu \nabla_\mu \phi \right) \left( \xi^\mu \nabla_\mu \phi \right) = \frac{1}{2} \omega (\omega - m\Omega_{\rm H}) \tilde{\phi}^2(x) \tag{2.3.43}$$

where we defined $\tilde{\phi}(x) \equiv \phi_0(r, \theta) \sin(\omega t - m\varphi)$. This quantity carried by the Killing field is invariant on the horizon. The relation (2.3.39) is thus given by

$$\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma = \frac{1}{2} \omega (\omega - m\Omega_{\rm H}) \int_{\mathcal{H}(r_+)} d\Sigma \tilde{\phi}^2(x). \tag{2.3.44}$$

We note that $d\Sigma = dAdv$ on the horizon where $A$ is the surface area of the horizon and the retarded time $v$ is an affine parameter on the horizon. The relation (2.3.44) generally diverges because of the integration with respect to $v$. We hence evaluate the energy current flux per unit time. The time averaged flux becomes

$$\int_{\mathscr{S}(\infty)} n_\mu J^\mu dA = -\int_{\mathcal{H}(r_+)} n_\mu J^\mu dA \tag{2.3.45}$$

$$= -\frac{1}{2}\omega(\omega - m\Omega_{\mathrm{H}}) \left|\tilde{\phi}_0\right|^2 . \tag{2.3.46}$$

where we defined $\left|\tilde{\phi}_0\right|^2 \equiv \int_{\mathcal{H}(r_+)} dA\tilde{\phi}^2(x)$. The right-hand side of (2.3.46) is positive for $\omega$ in the range

$$0 < \omega < m\Omega_{\mathrm{H}}. \tag{2.3.47}$$

Therefore we find that the outgoing energy current flux is larger than the incident one and the superradiance is present for the scalar field. The above discussion can be similarly performed for fermion fields. However, it is known that the right-hand side of (2.3.29) alway becomes zero and the superradiance is hence absent in the fermionic case [35, 36].

## 2.4 Dimensional Reduction near the Horizon

As stated in Section 2.1, the black hole uniqueness theorem is valid only in four dimensions and the Kerr-Newman solution is the most general solution in the 4-dimensional theory. The space-time outside the Kerr-Newman black hole is represented by the Kerr-Newman metric and its geometry becomes spherically asymmetric because of its own rotation. It is known that the 4-dimensional Kerr-Newman metric effectively becomes a 2-dimensional spherically symmetric metric by using the technique of the dimensional reduction near the horizon.

The essential idea is as follows: We consider the action for a scalar field. We can then ignore the mass, potential and interaction terms in the action because the kinetic term dominates in the high-energy theory near the horizon. By expanding the scalar field in terms of the spherical harmonics and using the above properties at horizon, we find that the integrand in the action dose not depend on angular variables. Thus we find that the 4-dimensional action with the Kerr-Newman metric effectively becomes a 2-dimensional action with a spherically symmetric metric.

In this section, we would like to discuss the dimensional reduction near the event horizon and actually show that the 4-dimensional Kerr-Newman metric effectively becomes a 2-dimensional spherically symmetric metric by using the technique of the dimensional reduction near the horizon.

For simplicity, we consider the 4-dimensional action for a complex scalar field

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu + ieV_\mu)\phi^* (\partial_\nu - ieV_\nu)\phi + S_{\text{int}}, \tag{2.4.1}$$

where the first term is the kinetic term and the second term $S_{\text{int}}$ represents the mass, potential and interaction terms. The gauge field $V_\mu$ associated with the Coulomb potential of the black hole, is given by

$$(V_\mu) = \left( -\frac{Qr}{r^2 + a^2}, 0, 0, 0 \right). \tag{2.4.2}$$

By substituting both the Kerr-Newman metric (2.1.8) and (2.4.2) to (2.4.1), we obtain

$$\begin{aligned} S = \int dt dr d\theta d\varphi \sin\theta \phi^* \Bigg[ & \left( \frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta \right) \left( \partial_t + \frac{ieQr}{r^2+a^2} \right)^2 \\ & + 2ia \left( \frac{r^2+a^2}{\Delta} - 1 \right) \left( \partial_t + \frac{ieQr}{r^2+a^2} \right) \hat{L}_z - \partial_r \Delta \partial_r + \hat{\boldsymbol{L}}^2 - \frac{a^2}{\Delta} \hat{L}_z^2 \Bigg] \phi + S_{\text{int}}, \end{aligned} \tag{2.4.3}$$

where we used

$$\hat{\boldsymbol{L}}^2 = -\frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta - \frac{1}{\sin^2\theta} \partial_\varphi^2, \tag{2.4.4}$$

$$\hat{L}_z = -i\partial_\varphi. \tag{2.4.5}$$

By performing the partial wave decomposition of $\phi$ in terms of the spherical harmonics

$$\phi = \sum_{l,m} \phi_{lm}(t,r) Y_{lm}(\theta, \varphi), \tag{2.4.6}$$

we obtain

$$\begin{aligned} S = \int dt dr d\theta d\varphi \sin\theta \sum_{l',m'} \phi^*_{l'm'} Y^*_{l'm'} \Bigg[ & \frac{(r^2+a^2)^2}{\Delta} \left( \partial_t + \frac{ieQr}{r^2+a^2} \right)^2 - a^2 \sin^2\theta \left( \partial_t + \frac{ieQr}{r^2+a^2} \right)^2 \\ & + 2ima \frac{r^2+a^2}{\Delta} \left( \partial_t + \frac{ieQr}{r^2+a^2} \right) - 2ima \left( \partial_t + \frac{ieQr}{r^2+a^2} \right) - \partial_r \Delta \partial_r + l(l+1) - \frac{m^2 a^2}{\Delta} \Bigg] \\ & \times \sum_{l,m} \phi_{lm} Y_{lm} + S_{\text{int}}, \end{aligned} \tag{2.4.7}$$

where we used eigenvalue equations for $\hat{\boldsymbol{L}}^2$ and $\hat{L}_z$

$$\hat{\boldsymbol{L}}^2 Y_{lm} = l(l+1) Y_{lm}, \tag{2.4.8}$$

$$\hat{L}_z Y_{lm} = m Y_{lm}. \tag{2.4.9}$$

Here $l$ is the azimuthal quantum number and $m$ is the magnetic quantum number. Now, we transform the radial coordinate $r$ into the tortoise coordinate $r_*$ defined by

$$\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta} \equiv \frac{1}{f(r)}. \tag{2.4.10}$$

After this transformation, the action (2.4.7) is written by

$$\begin{aligned} S = \int dtdr_* d\theta d\varphi \sin\theta \sum_{l',m'} \phi^*_{l'm'} Y^*_{l'm'} \Bigg[ &(r^2+a^2)\left(\partial_t + \frac{ieQr}{r^2+a^2}\right)^2 - f(r) a^2 \sin^2\theta \left(\partial_t + \frac{ieQr}{r^2+a^2}\right)^2 \\ &+ 2ima\left(\partial_t + \frac{ieQr}{r^2+a^2}\right) - F(r) 2ima \left(\partial_t + \frac{ieQr}{r^2+a^2}\right) - \partial_{r_*}(r^2+a^2)\partial_{r_*} \\ &+ f(r) l(l+1) - \frac{m^2a^2}{r^2+a^2}\Bigg] \sum_{l,m} \phi_{lm} Y_{lm} + S_{\text{int}}. \end{aligned} \tag{2.4.11}$$

Here we consider this action in the region near the horizon. Since $f(r_+) = 0$ at $r \to r_+$, we only retain dominant terms in (2.4.11). We thus obtain the effective action near the horizon $S_{(\mathrm{H})}$

$$\begin{aligned} S_{(\mathrm{H})} = \int dtdr_* d\theta d\varphi \sin\theta \sum_{l',m'} \phi^*_{l'm'} Y^*_{l'm'} \Bigg[ &(r^2+a^2)\left(\partial_t + \frac{ieQr}{r^2+a^2}\right)^2 + 2ima\left(\partial_t + \frac{ieQr}{r^2+a^2}\right) \\ &- \partial_{r_*}(r^2+a^2)\partial_{r_*} - \frac{m^2a^2}{r^2+a^2}\Bigg] \sum_{l,m} \phi_{lm} Y_{lm}, \end{aligned} \tag{2.4.12}$$

where we ignored $S_{\text{int}}$ by using $f(r_+) = 0$ at $r \to r_+$. Because the theory becomes the high-energy theory near the horizon and the kinetic term dominates, we can ignore all the terms in $S_{\text{int}}$. For example, we consider the case of a mass term. In this case, a mass term is usually given by

$$\int dx^4 \left(\mu^2 \phi^* \phi\right) = \int dtdrd\theta d\varphi \sin\theta \left(\mu^2 \phi^*\phi\right) \tag{2.4.13}$$

$$= \int dtdr_* d\theta d\varphi \sin\theta \left(f(r) \mu^2 \phi^* \phi\right), \tag{2.4.14}$$

where $\mu$ is a mass of the scalar field and we used (2.4.10) in the last line. We find that the term vanishes by using $f(r_+) = 0$ at $r \to r_+$. The same is equally true of other interaction terms $S_{\text{int}}$. After this analysis, we return to the expression written in terms of $r$, and we obtain

$$S_{(\mathrm{H})} = -\sum_{l,m} \int dtdr (r^2+a^2) \phi^*_{lm} \left[ -\frac{r^2+a^2}{\Delta}\left(\partial_t + \frac{ieQr}{r^2+a^2} + \frac{ima}{r^2+a^2}\right)^2 + \partial_r \frac{\Delta}{r^2+a^2} \partial_r \right] \phi_{lm}, \tag{2.4.15}$$

where we used the orthonormal condition for the spherical harmonics

$$\int d\theta d\varphi \sin\theta Y^*_{l'm'} Y_{lm} = \delta_{l',l} \delta_{m',m}. \tag{2.4.16}$$

From (2.4.15), we find that $\phi_{lm}$ can be considered as a (1+1)-dimensional complex scalar field in the backgrounds of the dilaton $\Phi$, metric $g_{\mu\nu}$ and two $U(1)$ gauge fields $V_\mu$, $U_\mu$

$$\Phi = r^2 + a^2, \tag{2.4.17}$$

$$g_{tt} = -f(r), \quad g_{rr} = \frac{1}{f(r)}, \quad g_{rt} = 0, \tag{2.4.18}$$

$$V_t = -\frac{Qr}{r^2 + a^2}, \quad V_r = 0, \tag{2.4.19}$$

$$U_t = -\frac{a}{r^2 + a^2}, \quad U_r = 0. \tag{2.4.20}$$

There are two $U(1)$ gauge fields: One is the original gauge field as in (2.4.2) while the other is the induced gauge field associated with the isometry along the $\varphi$ direction. The induced $U(1)$ charge of the 2-dimensional field $\phi_{lm}$ is given by $m$. Then the gauge potential $A_t$ is a sum of these two fields,

$$A_t \equiv eV_t + mU_t = -\frac{eQr}{r^2 + a^2} - \frac{ma}{r^2 + a^2}, \quad A_r = 0. \tag{2.4.21}$$

By using the above notations, the action (2.4.15) is rewritten as

$$S_{(\mathrm{H})} = -\sum_{l,m} \int dtdr\Phi\phi^*_{lm} \left[ g^{tt} \left(\partial_t - iA_t\right)^2 + \partial_r g^{rr} \partial_r \right] \phi_{lm}, \tag{2.4.22}$$

From (2.4.18), we find that the 4-dimensional spherically non-symmetric Kerr-Newman metric (2.1.8) effectively behaves as a 2-dimensional spherically symmetric metric in the region near the horizon only

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2. \tag{2.4.23}$$

For confirmation, we show how to derive the surface gravity on the horizon of the Kerr-Newman black hole from $f(r)$ defined by (2.4.10). Actually by calculating the surface gravity, we can obtain

$$\kappa_\pm \equiv \frac{1}{2} f'(r) \bigg|_{r=r_\pm} = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}, \tag{2.4.24}$$

where $\{'\}$ represents differentiation with respect to $r$. This result agrees with the well-known surface gravity on the horizon of the Kerr-Newman black hole as in (2.1.23).

## 2.5 Analogies between Black Hole Physics and Thermodynamics

To understand properties of black holes, it is very useful to understand the black hole physics in the context of generalized thermodynamics. The main reason is that there are various analogies

between the black hole physics and thermodynamics. It is said that the idea of making use of thermodynamic methods in black hole physics appears to have been first considered by Greif. He examined the possibility of defining the entropy of a black hole, but lacking many of the recent results in black hole physics, he did not make a concrete proposal [37]. Afterward, properties of black holes were analyzed by Bekenstein, Bardeen, Carter and Hawking and others, and analogies between black hole physics and thermodynamics were clarified [38,39]. The discussion is as follows.

By definition, a black hole can absorb matter but nothing, not even light, can classically escape from it. A black hole has a property that as a black hole absorbs matter, the black hole area increases. For example, we consider that two Schwarzschild black holes with masses $M_1$ and $M_2$ merge and then a black hole with a mass $M = M_1 + M_2$ is formed (Fig. 2.12). Before the merger, areas of two black holes are respectively $A_1 = 16\pi M_1^2$ and $A_2 = 16\pi M_2^2$. An area of the black hole after the merger is $A = 16\pi(M_1 + M_2)^2$. Compared between the sum of two black hole areas before the merger and the black hole area after the merger, we obtain an inequality for black hole areas

$$A_1 + A_2 \leq A. \tag{2.5.1}$$

This means that a black hole area after the merger is the same as the sum of each black hole area before the merger or is larger than it. A black hole area never classically decreases,

$$\delta A \geq 0, \tag{2.5.2}$$

since no black hole radiates matter or splits into any black holes. This result is known as Hawking's black hole area theorem [32].

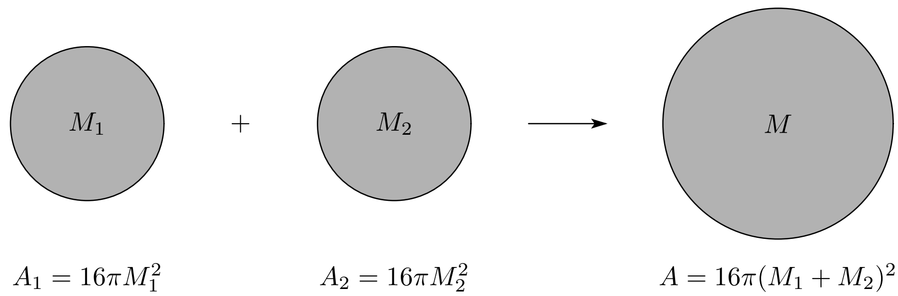


Fig. 2.12 The merger of black holes.

Here we would like to state the properties of entropy in thermodynamics. In thermodynamics, entropy represents the degree of concentration of matter and energy in a system. As a famous example of explaining entropy, we consider that one puts a drop of ink into a glass of water. At first, a drop of ink localizes in a certain part of the water. This is a state with low entropy. As time advances, a drop of ink distributes all over the water and the water achieve an even color at some time. This is that entropy is a state with higher entropy. The entropy of an isolated system $\mathcal{S}$ never decreases and rather increases over time, i.e.,

$$\delta\mathcal{S} \geq 0. \tag{2.5.3}$$

This is well-known as the second law of thermodynamics. Thus, both of the black hole area and entropy tend to increase irreversibly.

As with entropy, the black hole area is closely related to a degradation of energy, in other words, an unavailable energy. In thermodynamics, an increase of entropy means that a part of energy is unavailable, namely, the energy is no longer converted into work. There is the same relation in black hole physics. In Section 2.3, we showed that a part of energy can be extracted from a rotating black hole such as a Kerr black hole by the Penrose process. But all energy cannot be extracted from the black hole. The Kerr black hole gradually decreases the angular momentum by the Penrose process. When the black hole loses the total angular momentum, it becomes a Schwarzschild black hole. By the Hawking's black hole area theorem, the mass of the black hole is then larger than a mass of a Schwarzschild black hole obtained by taking $a = 0$ for the original Kerr black hole. This mass is called an *irreducible mass*. In the case of a Kerr-Newman black hole, the irreducible mass $M_{\rm ir}$ is given by

$$M_{\rm ir} = \sqrt{\frac{A}{16\pi}}. \tag{2.5.4}$$

It is regarded as an inactive energy which cannot be converted to work. The increase of an irreducible mass $M_{\rm ir}$, i.e., the increase of a black hole area $A$ thus corresponds to a degradation of the black hole energy in the thermodynamic sense.

As found from the above consideration, it is said that properties possessed by a black hole area $A$ are similar to ones possessed by the thermodynamic entropy and the Hawking's black hole area theorem corresponds to the second law of thermodynamics. Furthermore, by comparing the energy conservation law in the black hole physics with the first law of thermodynamics, we would like to clarify corresponding physical quantities in these two phenomena.

In general, the first law of thermodynamics is given by

$$d\mathcal{E} = \mathcal{T} d\mathcal{S} - d\mathcal{W}, \tag{2.5.5}$$

where $\mathcal{E}$ is the energy of the system, $\mathcal{T}$ is the temperature, $\mathcal{S}$ is the entropy and $\mathcal{W}$ is the work done by the system, while as already stated as in (2.1.22) of Section 2.1, the energy conservation law in black hole physics is given by

$$dM = \frac{\kappa}{8\pi} dA + \Omega_{\rm H} dL + \Phi_{\rm H} dQ, \tag{2.5.6}$$

Now we make comparisons between the relations (2.5.5) and (2.5.6). We make a table of the corresponding relationships between physical quantities of black hole physics and thermodynamics (Tab. 2.3). The correspondence relationship between the mass $M$ and the energy $\mathcal{E}$ in the left side of each relation is clear and it is well-known as the mass-energy equivalence by Einstein. The second term and the third term in (2.5.6) stand for work terms done by the rotation and the electromagnetism. It is considered that they correspond to the work term $-d\mathcal{W}$ done by the system in thermodynamics. We shall compare the remaining first term in each relation, i.e., $\dfrac{\kappa}{8\pi} dA$ and $\mathcal{T} d\mathcal{S}$. By making a black hole area correspond to entropy, we find that the surface gravity corresponds to the temperature in thermodynamics.

Tab. 2.3 The corresponding relationships between physical quantities of thermodynamics and black hole physics

| Thermodynamics | Black hole physics |
|---|---|
| Energy: $\mathcal{E}$ | Mass: $M$ |
| Temperature: $\mathcal{T}$ | Surface gravity: $\kappa$ |
| Entropy: $\mathcal{S}$ | Black hole area: $A$ |
| Work term done by system: $-d\mathcal{W}$ | Work terms done by rotation and electromagnetism: $\Omega_{\mathrm{H}} dL + \Phi_{\mathrm{H}} dQ$ |

Here we recall properties of both the surface gravity of a black hole and temperature. By definition, a surface gravity of the black hole represents the strength of the gravitational field on the event horizon. As found from (2.1.23), the surface gravity $\kappa$ is constant over the horizon in the stationary black hole. In thermal equilibrium, temperature also possesses the same property. It is well-known as the zeroth law of thermodynamics.

In passing, the third law of thermodynamics states that the temperature of the system cannot achieve the absolute zero temperature by a physical process. This is also called Nernst's theorem. It corresponds to the speculation that the surface gravity cannot achieve $\kappa = 0$ by a physical process in black hole physics. A reason for believing it is that if one could reduce it to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

From the above discussion, we find that the relationships between the laws of black hole physics and thermodynamics may be more than an analogy (Tab. 2.4, [40]). However, we do not know from only the above discussion that black holes actually have entropy and temperature. In the next section, we would like to discuss the consideration that black holes have entropy, which is due to Bekenstein.

Tab. 2.4 The corresponding relationships between the laws of thermodynamics and black hole physics

| Law | Thermodynamics | Black hole physics |
|---|---|---|
| Zeroth | $\mathcal{T}$ constant throughout body in thermal equilibrium | $\kappa$ constant over horizon of stationary black hole |
| First | $d\mathcal{E} = \mathcal{T} d\mathcal{S} - d\mathcal{W}$ | $dM = \dfrac{\kappa}{8\pi} dA + \Omega_{\mathrm{H}} dL + \Phi_{\mathrm{H}} dQ$ |
| Second | $\delta\mathcal{S} \geq 0$ in any process | $\delta A \geq 0$ in any process |
| Third | Impossible to achieve $\mathcal{T} = 0$ by a physical process | Impossible to achieve $\kappa = 0$ by a physical process |

## 2.6 Black Holes and Entropy

In 1973, Bekenstein proposed that a black hole has its entropy. He stated that the black hole entropy is represented by a function of the black hole area from the above analogies. If the black hole entropy is related to the black hole area, it has to satisfy the black hole area theorem. From this, he presumed that the black hole entropy is proportional to the black hole area.

Then, to find the proportionality coefficient, he considered that a particle with the least information falls into a black hole. When the particle falls into a black hole, the information of the particle is lost. In other words, it means an increase in the black hole entropy. He evaluated the proportionality coefficient by the conjecture that the black hole entropy equals the minimum area increased by dropping a matter into the black hole.

In this section, we would like to show the derivation of black hole entropy by Bekenstein. In Subsection 2.6.1, we simply explain the entropy of a particle with the least information in information theory. In Subsection 2.6.2, we explain that minimum entropy is increased when a particle falls into a black hole. In Subsection 2.6.3, we evaluate the black hole entropy by using assumptions that these two quantities are equal to each other.

### 2.6.1 Entropy in information theory

In physics, entropy represents a degree of concentration of matter and energies. The entropy is defined by Boltzmann's formula

$$\mathcal{S} = k \ln W, \tag{2.6.1}$$

where $W$ is the number of states and $k$ is Boltzmann's constant.

The connection between entropy and information is well-known [41, 42]. In information theory, the uncertain information and the missing information of the system are measured by the entropy. The probability of the $n$-th state in all known states of the system is defined as $P_n$ The entropy of the system is then defined by Shannon's formula

$$\mathcal{S} = -\sum_n P_n \ln P_n. \tag{2.6.2}$$

Here we note that entropy is dimensionless. We will present the discussion of dimensions later.

When a new piece of information is available for the system, we can find that probabilities $P_n$ are provided with some restrictions. For example, we consider the case of a die. The probabilities are respectively $\frac{1}{6}$ from 1 to 6. The entropy is then $\ln 6$ from (2.6.2). Now if we get new information that "There are odd numbers (or odd numbers are given)", then the probability of getting even numbers is zero, i.e., $P_2 = P_4 = P_6 = 0$. The probability of getting odd numbers is $\frac{1}{3}$ and thus the entropy is $\ln 3$. As found in the above discussion, as we get new information, the entropy locally decreases. This property is given by Brillouin's identification [43]

$$\Delta\mathcal{I} = -\Delta\mathcal{S}, \tag{2.6.3}$$

where $\Delta\mathcal{I}$ stands for the new information (bound information). This relation means that the bound information corresponds to the decrease of the entropy.

Here we would like to discuss the dimensions of the physical quantities. Entropy appearing in Boltzmann's formula (2.6.1) has the dimension of the energy divided by the temperature. Although it is possible to decide the dimension of information by using it, it is a custom in the information theory to treat information as a dimensionless quantity. Thus we adopt a unit system that both entropy and information is dimensionless. This means we select to measure temperature by the unit of energy. Then Boltzmann's constant is also dimensionless. By adopting the above unit system, the equations (2.6.2) and (2.6.3) are satisfied as dimensionless quantities.

The conventional unit of information is the "bit" which may be defined as the information available when the answer to a yes-or-no question is precisely known, i.e., the entropy is zero. Of course, the unit is dimensionless. According to (2.6.3), a bit is also numerically equal to the maximum entropy that can be associated with a yes-or-no question, i.e., the entropy when no information whatsoever is available about the answer. From (2.6.2), the entropy in the yes-or-no question is written as

$$\mathcal{S} = -P_{\text{yes}} \ln P_{\text{yes}} - P_{\text{no}} \ln P_{\text{no}} \tag{2.6.4}$$

$$= -P_{\text{yes}} \ln P_{\text{yes}} - (1 - P_{\text{yes}}) \ln(1 - P_{\text{yes}}). \tag{2.6.5}$$

We thus find that the entropy is the maximum value $\ln 2$ when $P_{\text{yes}} = P_{\text{no}} = \dfrac{1}{2}$ and one bit is equal to $\ln 2$ of information.

Let us now return to our original subject, black hole. We consider that a particle falls into a black hole. An amount of information of the particle would depend on how much is known about the internal states of the particle. The minimum information loss for the particle would be contained in the answer to the question "Does the particle exist or not?" Before the particle drops into the black hole, the answer is known to be "yes". But after the particle drops into the black hole, we have no information whatever about the answer. This is because one knows nothing about the physical conditions inside the black hole, and thus one cannot assess the likelihood of the particle continuing to exist or being destroyed. One must, therefore, admit the loss of one bit of information at the very least. This means that the entropy is increased by

$$\Delta\mathcal{S} = \ln 2, \tag{2.6.6}$$

before and after the particle with the tiniest information falls into the black hole.

### 2.6.2 The minimum increase of the black hole area

In this subsection, we calculate the minimum possible increase in the black hole area, which must result when a spherical particle of rest mass $\mu$ and proper radius $b$ is captured by a Kerr-Newman black hole. Bekenstein used the "rationalized area" of a black hole $\alpha$ defined by

$$\alpha \equiv \frac{A}{4\pi}, \tag{2.6.7}$$

where $A$ is the black hole area as in (2.1.20). The first law of black hole physics (2.1.22) is then written as

$$dM = \Theta_{\text{H}} d\alpha + \Omega_{\text{H}} dL + \Phi_{\text{H}} dQ, \tag{2.6.8}$$

where $\Theta_{\rm H}$ is defined by

$$\Theta_{\rm H} \equiv \frac{r_+ - M}{2\alpha}. \tag{2.6.9}$$

There are several ways in which a particle may fall into a black hole. All these bring the increase of the black hole area. We are interested in the method for inserting the particle which results in the smallest increase. This method has already been discussed by Christodoulou in connection with his introduction of the concept of irreducible mass [31, 44]. The essence of Christodoulou's method is that if a freely falling point particle is captured by a Kerr-Newman black hole, then the irreducible mass and, consequently, the area of the black hole is left unchanged. Bekenstein generalized Christodoulou's method to a particle with a proper radius and showed the increased area of the black hole is no longer precisely zero when the particle falls into the black hole.

We assume that a freely falling particle is neutral. The trajectory of the particle follows a geodesic of the Kerr-Newman metric (2.1.8). The horizon is located at $r = r_+$ where $r_\pm$ are defined by (2.1.17).

First integrals for geodesic motion in the Kerr-Newman background have been given by Carter [45]. Christodoulou used the first integral

$$E^2[r^4+a^2(r^2+2Mr-Q^2)]-2E(2Mr-Q^2)ap_\varphi-(r^2-2Mr+Q^2)p_\varphi^2-(\mu^2r^2+q)\Delta = (p_r\Delta)^2, \tag{2.6.10}$$

as a starting point of his analysis. We show the derivation in Appendix B. In (2.6.10), $E = -p_t$ is the conserved energy, $p_\varphi$ is the conserved component of angular momentum in the direction of the axis of symmetry, $q$ is Carter's fourth constant of the motion, $\mu$ is the rest mass of the particle and $p_r$ is its covariant radial momentum.

By following Christodoulou, we solve (2.6.10) for $E$:

$$E = \mathcal{B}ap_\varphi + \sqrt{\left(\mathcal{B}^2a^2 + \frac{r^2-2Mr+Q^2}{\mathcal{A}}\right)p_\varphi^2 + \frac{(\mu^2r^2+q)\Delta + (p_r\Delta)^2}{\mathcal{A}}}, \tag{2.6.11}$$

where

$$\mathcal{A} \equiv r^4 + a^2(r^2+2Mr-Q^2), \tag{2.6.12}$$

$$\mathcal{B} \equiv \frac{(2Mr-Q^2)}{\mathcal{A}}. \tag{2.6.13}$$

The definitions (2.6.12) and (2.6.13) at the horizon as in (2.1.16) are written as

$$\mathcal{A}(r=r_+) \equiv \mathcal{A}_+ = (r_+^2+a^2)^2, \tag{2.6.14}$$

$$\mathcal{B}(r=r_+) \equiv \mathcal{B}_+ = \frac{1}{r_+^2+a^2}. \tag{2.6.15}$$

Furthermore, at the horizon, we obtain

$$\mathcal{B}_+ a = \Omega_{\rm H}. \tag{2.6.16}$$

where $\Omega_{\rm H}$ is defined by (2.1.24). The coefficient of $p_\varphi^2$ at the horizon vanishes

$$\mathcal{B}_+^2 a^2 + \frac{r_+^2 - 2Mr_+ + Q^2}{\mathcal{A}_+} = \frac{a^2}{(r_+^2 + a^2)^2} + \frac{r_+^2 - 2Mr_+ + Q^2}{(r_+^2 + a^2)^2} = \frac{\Delta}{(r_+^2 + a^2)^2} = 0, \tag{2.6.17}$$

and the coefficient of $\mu^2 r^2 + q$ also vanishes. However, since $p_r\Delta$ cannot be defined at the horizon because of $p_r = g_{rr}p^r$, we retain the term as

$$p_r = \frac{\Sigma}{\Delta} p^r \tag{2.6.18}$$

$$\Leftrightarrow \quad p_r\Delta = (r^2 + a^2\cos^2\theta)p^r. \tag{2.6.19}$$

If the particle's orbit intersects the horizon, we then have from (2.6.11) that

$$E = \Omega_{\rm H} p_\varphi + \frac{|p_r\Delta|_+}{\sqrt{\mathcal{A}_+}}. \tag{2.6.20}$$

As a result of the capture, the mass of the black hole increases by $E$ and its component of the angular momentum in the direction of the symmetry axis increases by $p_\varphi$. By comparing (2.1.22) with (2.6.20), the black hole's rationalized area $\alpha$ increases by $\dfrac{|p_r\Delta|_+}{\Theta_{\rm H}\sqrt{\mathcal{A}_+}}$. As pointed out by Christodoulou, by taking

$$|p_r\Delta|_+ = 0, \tag{2.6.21}$$

the relation (2.6.20) becomes

$$E = \Omega_{\rm H} p_\varphi, \tag{2.6.22}$$

and the increase of the black hole area vanishes. The above analysis shows that it is possible for a black hole to capture a point particle without increasing its area.

Here, by following Bekenstein's extension, we would like to show how this conclusion is changed if the particle has a nonzero proper radius $b$. The relation (2.6.11) always describes the motion of the particle's center of mass at the moment of capture. It should be clear that to generalize Christodoulou's result to the present case one should evaluate (2.6.11) not at $r = r_+$, but $r = r_+ + \delta$, where $\delta$ is determined by

$$\int_{r_+}^{r_+ + \delta} \sqrt{g_{rr}}\, dr = b. \tag{2.6.23}$$

$r = r_+ + \delta$ is a point a proper distance $b$ outside the horizon. By using the component $g_{rr}$ as in (2.1.15), we find

$$b = 2\sqrt{\frac{\delta(r_+^2 + a^2\cos^2\theta)}{r_+ - r_-}}, \tag{2.6.24}$$

where we assumed that $r_+ - r_- \gg \delta$. Expanding the argument of the square root in (2.6.11) in powers of $\delta$, replacing $\delta$ by its value given by (2.6.24), and keeping only terms to $O(b)$ we obtain

$$E = \Omega_{\rm H} p_\varphi + \sqrt{\left(\frac{r_+^2 - a^2}{r_+^2 + a^2}\right) p_\varphi^2 + \mu^2 r_+^2 + q} \times \frac{1}{2} b \frac{r_+ - r_-}{(r_+^2 + a^2)} \times \frac{1}{\sqrt{r_+^2 + a^2\cos^2\theta}}. \tag{2.6.25}$$

This relation (2.6.25) is the generalization to $O(b)$ of Christodoulou's result (2.6.22). Carter's kinetic constant $q$ is given by

$$q = \cos^2\theta \left[a^2(\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2\theta}\right] + p_\theta^2 \tag{2.6.26}$$

This constant appeared in the derivation of (2.6.10) (see Appendix B). We can obtain a lower bound for it as follows. From the requirement that the $\theta$ momentum $p_\theta$ is real in (2.6.26), we obtain

$$q \geq \cos^2\theta \left[a^2(\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2\theta}\right], \tag{2.6.27}$$

where the equality holds when $p_\theta = 0$. If we replace $E$ in (2.6.27) by $\Omega_{\rm H} p_\varphi$ as in (2.6.22), we obtain

$$q \geq \cos^2\theta \left[a^2\mu^2 + p_\varphi^2\left(\frac{1}{\sin^2\theta} - a^2\Omega_{\rm H}^2\right)\right]. \tag{2.6.28}$$

We know that $\dfrac{1}{\sin^2\theta} \geq 1$ and it is easily shown that $a^2\Omega_{\rm H}^2 \leq \dfrac{1}{4}$ for a Kerr-Newman black hole. Since the coefficient of $p_\varphi^2$ is positive, we can take the constant $q$ as a smaller value

$$q \geq a^2\mu^2\cos^2\theta, \tag{2.6.29}$$

when $p_\varphi = 0$. By substituting (2.6.29) into (2.6.25), we obtain

$$E \geq \Omega_{\rm H} p_\varphi + \frac{1}{2}\mu b \frac{r_+ - r_-}{r_+^2 + a^2}, \tag{2.6.30}$$

where the equality holds when $p_\varphi = p_\theta = p^r = 0$. This relation is correct to $O(b)$. The increase in the rationalized area of the black hole, computed by means of (2.6.8), (2.6.9) and (2.6.30), is given by

$$\Delta\alpha \geq 2\mu b. \tag{2.6.31}$$

This gives the fundamental lower bound on the increase in the rationalized area of the black hole $\Delta\alpha$,

$$(\Delta\alpha)_{\min} = 2\mu b. \tag{2.6.32}$$

We note that it is independent of $M$, $Q$ and $L$.

We can make $(\Delta\alpha)_{\min}$ smaller by making $b$ smaller. However, we must remember that $b$ can be no smaller than the particle's Compton wavelength $\frac{\hbar}{\mu}$, or the Schwarzschild radius $2\mu$. If the Compton wavelength is larger than the Schwarzschild radius $\frac{\hbar}{\mu} \geq 2\mu$, namely, the mass of the particle satisfies $\mu \leq \sqrt{\frac{\hbar}{2}}$, we can make $b$ smaller to $\frac{\hbar}{\mu}$. If the Schwarzschild radius is larger than the Compton wavelength $\frac{\hbar}{\mu} < 2\mu$, namely, the mass of the particle satisfies $\mu > \sqrt{\frac{\hbar}{2}}$, we can make $b$ smaller to $2\mu$. The relation (2.6.32) is thus given by $2\hbar$, when $b = \frac{\hbar}{\mu}$, and given by $4\mu^2$, when $b \simeq 2\mu$. Since $4\mu^2 > 2\hbar$, we can determine a lower bound of the rationalized area of a Kerr-Newman black hole as

$$(\Delta\alpha)_{\min} = 2\hbar, \tag{2.6.33}$$

when the black hole captures the particle.

### 2.6.3 Information loss and black hole entropy

In Section 2.5, we already stated that a black hole area is similar to the entropy in thermodynamics. Although there are clear analogies between them, we do not know how to identify the black hole area as the black hole entropy. In this subsection, we would like to present the discussion by Bekenstein [38].

To begin with, we consider that a black hole is formed by the gravitational collapse of a very heavy star. According to the no-hair theorem [16], the stationary state of the black hole is completely characterized by three parameters, i.e., the mass, the angular momentum and the charge. Thus black holes do not depend on the internal configuration of the collapsed body. This means that a lot of information is lost by the gravitational collapse. It is then natural to introduce the concept of black hole entropy as the measure of the inaccessibility of information to an exterior observer. Furthermore, we consider that the black hole entropy is associated with the black hole area.

Bekenstein assumed that the entropy of a black hole $\mathcal{S}_{\rm BH}$ is some monotonically increasing function of its rationalized area as in (2.6.7):

$$\mathcal{S}_{\rm BH} = f(\alpha). \tag{2.6.34}$$

The entropy of an evolving thermodynamic system increases due to the gradual loss of information which is a consequence of the washing out of the most of the initial conditions. Now, as a black hole approaches equilibrium, the effects of the initial conditions are also washed out (the black hole loses its hair). One would thus expect that the loss of information about initial peculiarities of the black hole will be reflected in a gradual increase in $\mathcal{S}_{\rm BH}$. Indeed the relation (2.6.34) predicts just this.

One possible choice for $f$ in (2.6.34),

$$f(\alpha) \propto \sqrt{\alpha}, \tag{2.6.35}$$

is untenable on some reasons. We consider two black holes which start off very distant from each other. Since they interact weakly, we can take the total black hole entropy to be the sum of $\mathcal{S}_{\rm BH}$ of each black hole. The black holes now move closer together and finally merge, and form a black hole which settles down to equilibrium. In the process no information about the black hole interior can become available. On the contrary, much information is lost as the final black hole loses its hair. Thus, we expect the final black hole entropy to exceed the initial one. By the assumption (2.6.35), this implies that the irreducible mass (2.5.4) of the final black hole exceeds the sum of irreducible masses of the initial black holes. Now suppose that all three black holes are Schwarzschild ($M = M_{\rm ir}$). We are then confronted with the prediction that the final black hole mass exceeds the initial one. But this is nonsensical since the total black hole mass can only decrease due to gravitational radiation losses. We thus see that the choice as in (2.6.35) is untenable.

The next simplest choice for $f$ is

$$f(\alpha) = \gamma\alpha, \tag{2.6.36}$$

where $\gamma$ is a constant. Repetition of the above argument for this new $f$ leads to the conclusion that the final black-hole area must exceed the total initial black hole area. But we know this to be true from Hawking's theorem [32]. Thus the choice (2.6.36) leads to no contradiction. Therefore, Bekenstein adopted (2.6.36) for the moment.

Comparison of (2.6.35) and (2.6.36) shows that $\gamma$ must have the units of $[\text{length}]^{-2}$. But there is no constant with such units in classical general relativity. And so Bekenstein found only one truly universal constant $\hbar^{-1}$ with the correct units, where $\hbar$ is the Planck constant. Until now, although we used the natural system of units (1.0.1), we shall clearly describe the constant $\hbar$ in this subsection. Thus Bekenstein represented (2.6.34) as

$$\mathcal{S}_{\mathrm{BH}} = \frac{\eta\alpha}{\hbar}, \tag{2.6.37}$$

where $\eta$ is a dimensionless constant. This expression was also proposed by Bekenstein earlier from a different point of view [46]. It is well known that the Planck constant $\hbar$ also appears in the formulas for the entropy in thermodynamics, for example, the Sackur-Tetrode equation (see, for example, [47]).

To determine the value of $\eta$, Bekenstein considered that a particle falls into a Kerr-Newman black hole. In Subsection 2.6.1, we showed that the loss of one bit of information before and after the particle with the least information falls into a black hole, i.e., the increased entropy is $\Delta\mathcal{S} = \ln 2$. In Subsection 2.6.2, we showed that when a spherical particle with a radius, which is as large as the Compton wavelength, falls into a black hole, the minimum increase of the black hole area is given by (2.6.32). From (2.6.32), we obtain the increase of black hole entropy given by

$$(\Delta\mathcal{S}_{\mathrm{BH}})_{\min} = 2\hbar\frac{df(\alpha)}{d\alpha}. \tag{2.6.38}$$

Bekenstein conjectured that this entropy agrees with the loss of one bit of information (2.6.6) (see Fig. 2.13). We thus obtain

$$2\hbar\frac{df(\alpha)}{d\alpha} = \ln 2. \tag{2.6.39}$$

In the left-hand side of (2.6.39), the limit as in (2.6.33) can be attained only for a particle whose dimension is given by its Compton wavelength. Only such an “elementary particle” may be regarded as having no internal structure. We can thus consider that the loss of information associated with the loss of such a particle should be minimum. By integrating (2.6.39) over $\alpha$, we obtain

$$f(\alpha) = \left(\frac{1}{2}\ln 2\right)\frac{\alpha}{\hbar}. \tag{2.6.40}$$

From (2.6.34), we can obtain the black hole entropy

$$\mathcal{S}_{\mathrm{BH}} = \left(\frac{1}{2}\ln 2\right)\frac{\alpha}{\hbar}. \tag{2.6.41}$$

This form agrees with that of (2.6.37).

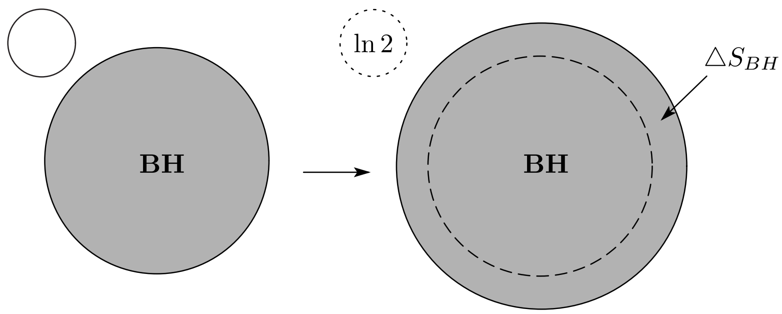


Fig. 2.13 Information loss and increase of entropy

Bekenstein showed the dependence of the black hole entropy $\mathcal{S}_{\rm BH}$ on the black hole area $\alpha$ from the above discussion, and the black hole entropy is given by

$$\mathcal{S}_{\rm BH} = \frac{\ln 2}{8\pi}\frac{kc^3}{\hbar G}A, \tag{2.6.42}$$

where we write the conventional units explicitly. However, he used some conjectures, and the relation $\eta = \frac{1}{2}\ln 2$ is derived by a certain assumption. The assumption is that the smallest possible radius of a particle is precisely equal to its Compton wavelength whereas the actual radius is not so sharply defined. Furthermore, an amount of information of such a particle might be more than $\ln 2$ because the particle has information for the mass and the radius. According to the current understanding, the black hole entropy is given by

$$\mathcal{S}_{\rm BH} = \frac{1}{4}\frac{kc^3}{\hbar G}A. \tag{2.6.43}$$

In comparison between (2.6.42) and (2.6.43), the value of $\eta$ is slightly different. However, Bekenstein had stated in his paper [38] that it would be somewhat pretentious to attempt to calculate the precise value of the constant $\dfrac{\eta}{\hbar}$ without a full understanding of the quantum reality which underlies a "classical" black hole. Surprisingly, he already suggested that the derivation of black hole radiation needs the consideration of quantum theory.

Bekenstein also defined a characteristic temperature for a Kerr-Newman black hole by

$$\frac{1}{\mathcal{T}_{\rm BH}} = \left(\frac{\partial \mathcal{S}_{\rm BH}}{\partial M}\right)_{L,Q}, \tag{2.6.44}$$

which is an analogue of the thermodynamic relation

$$\frac{1}{\mathcal{T}} = \left(\frac{\partial \mathcal{S}}{\partial \mathcal{E}}\right)_{V}. \tag{2.6.45}$$

By using both (2.6.8) and (2.6.9) in (2.6.44), we can obtain

$$\mathcal{T} = \frac{2\hbar}{\ln 2}\Theta_{\mathrm{H}}. \tag{2.6.46}$$

But Bekenstein did not regard this temperature as the temperature of the black hole. Because if a black hole has a temperature, some radiation from the black hole may appear. This conflicts with the classical definition. By definition, a black hole can only absorb matter but cannot radiate matter. Bekenstein did not suggest that a black hole has a temperature for the above reason.

# Chapter 3

# Black Hole Radiation

In the classical theory, black holes can only absorb matter but cannot radiate matter. Bekenstein proposed that a black hole has entropy from the point of view of information theory but could not suggest that a black hole has a temperature from his reasoning. Therefore, the complete correspondence between black hole physics and thermodynamics cannot be obtained. However, Hawking showed that a black hole continuously radiates its energy by taking quantum effects into account [2]. Furthermore, it was found that a black hole behaves as the a black body with a certain temperature and continuously performs its radiation. This is consistent with the classical definition of black holes. This black hole radiation is commonly called Hawking radiation.

The intuitive explanation is as follows [48]: According to quantum field theory, it is considered that a particle-antiparticle pair is formed by a fluctuation of energy everywhere in our universe. The antiparticle (negative energy state) formed by the pair creation can exist only for a very short time since it is unstable in our universe. The particle-antiparticle pair therefore vanishes by the pair annihilation after a certain short period of time.

Now we consider a pair creation very close to the event horizon outside a black hole. The pair creation arises in globally curved space-time because general relativity is based on the assumption that the space-time can be made locally flat. If a particle-antiparticle pair is formed very close to the horizon, the antiparticle (negative energy state) can fall into the black hole through the horizon in a certain short period of time. It is possible to show that the antiparticle can be put into a realizable orbit inside the event horizon. For an external observer, the black hole decreases its energy by absorbing the negative energy (antiparticle), while the particle with the positive energy, which is the same amount as the decreased energy of the black hole, can escape to infinity, since it can stably exist in our universe (Fig. 3.1). Therefore, we can understand that black holes radiate

particles. This is the mechanism of Hawking radiation.

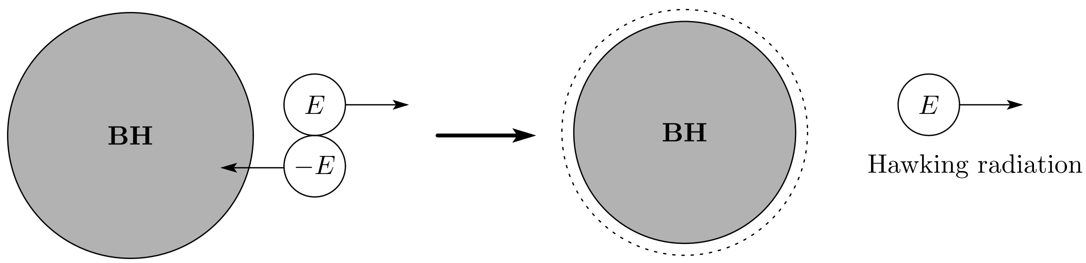


Fig. 3.1 Hawking mechanism

In this chapter, we would like to present several previous works on Hawking radiation. For sake of simplicity, we consider the case of a Schwarzschild black hole. The contents of this chapter are as follows. In Section 3.1, we review Hawking's original derivation of Hawking radiation. In Section 3.2, we would like to discuss some representative derivations of Hawking radiation briefly.

## 3.1 Hawking's Original Derivation

Hawking showed that black holes radiate matter by using quantum field theory in black hole physics. In this section, we would like to show the original derivation of Hawking radiation by Hawking [2].

For sake of simplicity, we consider a free massless scalar field. In Minkowski space, it satisfies the Klein-Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0, \tag{3.1.1}$$

where $\phi$ is a massless hermitian scalar field, $\eta^{\mu\nu}$ is the Minkowski metric (2.1.12) and $\partial_\mu$ is the partial derivative defined by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{3.1.2}$$

The ordinary derivative of $\phi$ is also written as $\phi_{,\mu}$. We can decompose the field into positive and negative frequency components

$$\phi = \sum_i (\varphi_i \boldsymbol{a}_i + \varphi_i^* \boldsymbol{a}_i^\dagger), \tag{3.1.3}$$

where $\{\varphi_i\}$ are a complete orthonormal family of complex valued solutions of the wave equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0. \tag{3.1.4}$$

It contains only positive frequencies with respect to the usual Minkowski time coordinate. The operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ are interpreted as the annihilation and creation operators respectively for particles in the $i$-th state. The vacuum state $|0\rangle$ is defined to be the state from which one cannot annihilate any particle, i.e.,

$$\boldsymbol{a}_i|0\rangle = 0, \qquad \text{for all } i. \tag{3.1.5}$$

The orthonormal condition is then given by

$$\rho_M(\varphi_i, \varphi_j) \equiv \frac{1}{2}i\int_V (\varphi_i\partial_t\varphi_j^* - \varphi_j^*\partial_t\varphi_i)dx^3 = \delta_{ij}, \tag{3.1.6}$$

where $V$ is a suitable closed space.

We considered quantum field theory in Minkowski space so far. Here, we would like to extend Minkowski space-time to curved space-time which is produced by the intense gravity of a black hole. In the curved space-time, the metric changes from the Minkowski metric to the metric of the curved space-time. Also physical laws must hold in any coordinate system. The partial derivatives contained in these laws must be replaced by the covariant derivatives in the curved space-time [49]. The covariant derivative is commonly represented by

$$\nabla_\mu \equiv \phi_{;\mu}. \tag{3.1.7}$$

The covariant derivative for a scalar field $\phi$ is given by

$$\nabla_\mu\phi = \partial_\mu\phi, \tag{3.1.8}$$

and the covariant derivative for a vector field $A_\nu$ is given by

$$\nabla_\mu A_\nu = \partial_\mu A_\nu + \Gamma^\alpha_{\nu\mu}A_\alpha, \tag{3.1.9}$$

where $\Gamma^\alpha_{\nu\mu}$ is the Christoffel symbol defined by (2.1.5). In curved space-time, the Klein-Gordon equation for a massless hermitian scalar field is thus represented by

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0. \tag{3.1.10}$$

We can use the relation (3.1.3) in the flat space. However, we cannot decompose the field into positive and negative frequency components in curved space. One can still require that the $\{f_i\}$ and the $\{f_i^*\}$ together form a complete the basis for solutions of the wave equations with

$$\rho(\varphi_i, \varphi_j) = -\frac{1}{2} i \int_\Sigma (\varphi_i \nabla_\mu \varphi_j^* - \varphi_j^* \nabla_\mu \varphi_i) d\Sigma^\mu = \delta_{ij}, \tag{3.1.11}$$

where $d\Sigma$ stands for an area element and $\Sigma$ is called a Cauchy surface which represents a suitable surface.

Here we recall the Penrose diagram drawn in Fig. 2.7. In the past null infinity $\mathcal{J}^-$, the Schwarzschild metric is asymptotically flat (the Minkowski metric) since $r \to \infty$. We can thus expand the field operator $\phi$ which satisfies the Klein-Gordon equation (3.1.10) as

$$\phi = \sum_i \{f_i \boldsymbol{a}_i + f_i^* \boldsymbol{a}_i^\dagger\}, \tag{3.1.12}$$

where $\{f_i\}$ is a family of solutions of the wave equation

$$\eta^{\mu\nu} \nabla_\mu \nabla_\nu f_i = 0. \tag{3.1.13}$$

In a manner similar to (3.1.11), they satisfy the orthonormal condition at $\mathcal{J}^-$

$$\rho(f_i, f_j^*) = \frac{1}{2} i \int_\Sigma (f_i \nabla_\mu f_j^* - f_j^* \nabla_\mu f_i) d\Sigma^\mu = \delta_{ij}, \tag{3.1.14}$$

where we note that $\{f_i\}$ only contain positive frequencies with respect to the canonical affine parameter on $\mathcal{J}^-$. It is natural that the operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ are respectively regarded as the annihilation and creation operators at $\mathcal{J}^-$. The vacuum at $\mathcal{J}^-$ is thus defined by

$$\boldsymbol{a}_i |0_-\rangle = 0. \tag{3.1.15}$$

Similarly, in the future null infinity $\mathcal{J}^+$, the Schwarzschild metric is asymptotically flat since $r \to \infty$. We can also expand the field operator $\phi$ by

$$\phi = \sum_i \left\{ p_i \boldsymbol{b}_i + p_i^* \boldsymbol{b}_i^\dagger + q_i \boldsymbol{c}_i + q_i^* \boldsymbol{c}_i^\dagger \right\}, \tag{3.1.16}$$

where $\{p_i\}$ are solutions of the wave equation which can escape to $\mathcal{J}^+$ and $\{q_i\}$ are solutions of the wave equation which cannot escape to $\mathcal{J}^+$ since they are absorbed by the future event horizon $\mathcal{H}^+$, namely, $\{p_i\}$ are zero at $\mathcal{H}^+$ and $\{q_i\}$ are zero at $\mathcal{J}^+$. The operators $\boldsymbol{b}_i$ and $\boldsymbol{b}_i^\dagger$ respectively stand for the annihilation and creation operators at $\mathcal{J}^+$, and the operators $\boldsymbol{c}_i$ and $\boldsymbol{c}_i^\dagger$ respectively stand

for the annihilation and creation operators at $\mathcal{H}^+$. The vacua at $\mathcal{J}^+$ and $\mathcal{H}^+$ are thus defined by

$$\boldsymbol{b}_i|0_+\rangle = 0, \tag{3.1.17}$$

$$\boldsymbol{c}_i|0_{\mathcal{H}^+}\rangle = 0, \tag{3.1.18}$$

where we also note that $\{p_i\}$ contain positive frequencies only with respect to the canonical affine parameter on $\mathcal{J}^+$. Although it is not clear whether one should impose some positive frequency condition on $\{q_i\}$, we would like to consider particles which start from $\mathcal{J}^-$, pass through the collapsing body and can escape to $\mathcal{J}^+$. The choice of the $\{q_i\}$ does not affect the calculation of the emission of particle to $\mathcal{J}^+$ since the $\{q_i\}$ are zero at $\mathcal{J}^+$. We require that $\{p_i\}$ and $\{p_i^*\}$ are a complete orthonormal family which satisfies

$$\rho'(p_i, p_j^*) = \frac{1}{2} i \int_{\Sigma'} (p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i) d\Sigma'^\mu = \delta_{ij}. \tag{3.1.19}$$

Here we would like to show that the relation (3.1.19) is satisfied even if one uses $\Sigma$ which appeared in (3.1.14) instead of $\Sigma'$. We thus consider $\rho(p_i, p_j^*) - \rho'(p_i, p_j^*)$. If the stable surface $\Sigma'$ differs from $\Sigma$, $\Sigma'$ can smoothly intersect with $\Sigma$ at certain points since $\Sigma'$ is not parallel to $\Sigma$. We represent the 4-dimensional volume enclosed by these two surfaces as $V$ (Fig. 3.2). By using the 4-dimensional Gauss theorem, we obtain

$$\rho(p_i, p_j^*) - \rho'(p_i, p_j^*) = \int_V d^4x \sqrt{-g} \nabla^\mu (p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i), \tag{3.1.20}$$

where $g$ stands for the determinant of the metric $g_{\mu\nu}$ defined by

$$g \equiv \det\left(g_{\mu\nu}\right), \tag{3.1.21}$$

namely, $\sqrt{-g}$ stands for the Jacobian with respect to the transformation from $d^4x$ to $d\Sigma$. By calculating the integrand of the right-hand side in (3.1.20), we obtain

$$\begin{aligned} \nabla^\mu (p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i) =& \nabla^\mu p_i \nabla_\mu p_j^* + p_i \nabla^\mu \nabla_\mu p_j^* - \nabla^\mu p_j^* \nabla_\mu p_i - p_j^* \nabla^\mu \nabla_\mu p_i \\ =& p_i \nabla^\mu \nabla_\mu p_j^* - p_j^* \nabla^\mu \nabla_\mu p_i. \end{aligned} \tag{3.1.22}$$

By using the Klein-Gordon equation (3.1.10), the relation (3.1.22) becomes

$$\nabla^\mu (p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i) = 0. \tag{3.1.23}$$

Since the right-hand side of (3.1.19) is zero, we obtain

$$\rho(p_i, p_j^*) = \rho'(p_i, p_j^*). \tag{3.1.24}$$

This means that $\rho(p_i, p_j^*)$ does not depend on $\Sigma$. Namely, if the Gauss theorem is satisfied, it means that we can freely choose the surface $\Sigma$ in (3.1.11)

$$\rho(p_i, p_j^*) = \frac{1}{2} i \int_\Sigma (p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i) d\Sigma^\mu = \delta_{ij}. \tag{3.1.25}$$

It is known that the above discussion is also valid for a scalar field with a mass [50].

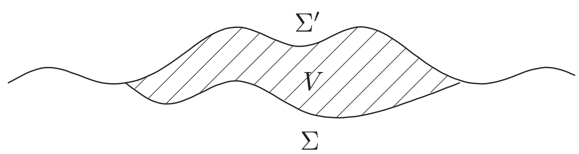


Fig. 3.2 The volume $V$ enclosed by two surfaces $\Sigma$ and $\Sigma'$

In the transitional time between $\{f_i\}$ and $\{p_i\}$, a collapsing body will appear. We do not know the corresponding solutions since we do not know the metric inside this region. By using the analogy of the tunneling effect, we can represent $\{p_i\}$ which appear at $\mathcal{J}^+$ as the linear combinations of $\{f_i\}$ with $\{f_i^*\}$

$$p_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*), \tag{3.1.26}$$

where $\alpha_{ij}$ and $\beta_{ij}$ are proportionality coefficients which stand for the amplitude ratio (Fig. 3.3). We substitute (3.1.26) into (3.1.16). Since $\{q_i\} = 0$ at $\mathcal{J}^+$, the relation (3.1.16) becomes

$$\phi = \sum_i \left\{ \sum_j (\boldsymbol{b}_j \alpha_{ij} + \boldsymbol{b}_j^\dagger \beta_{ij}^*) f_i + \sum_j (\boldsymbol{b}_j \beta_{ij} + \boldsymbol{b}_j^\dagger \alpha_{ij}^*) f_i^* \right\}. \tag{3.1.27}$$

By comparison between (3.1.12) and (3.1.27), we obtain

$$\boldsymbol{a}_i = \sum_j (\boldsymbol{b}_j \alpha_{ij} + \boldsymbol{b}_j^\dagger \beta_{ij}^*), \tag{3.1.28}$$

$$\boldsymbol{a}_i^\dagger = \sum_j (\boldsymbol{b}_j \beta_{ij} + \boldsymbol{b}_j^\dagger \alpha_{ij}^*). \tag{3.1.29}$$

We also find that the inverse transformations with respect to $\boldsymbol{b}_j$ and $\boldsymbol{b}_j^\dagger$ are obtained by

$$\boldsymbol{b}_i = \sum_j (\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger), \tag{3.1.30}$$

$$\boldsymbol{b}_i^\dagger = \sum_j (\alpha_{ij} \boldsymbol{a}_j^\dagger - \beta_{ij} \boldsymbol{a}_j). \tag{3.1.31}$$

These transformations are called the Bogoliubov transformations. For details of this calculation, see Appendix C.

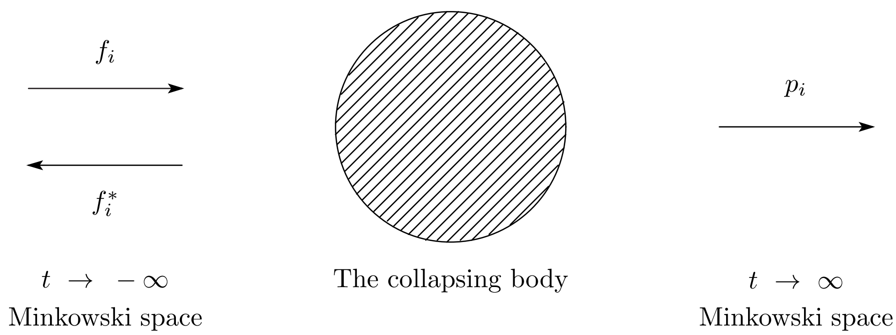


Fig. 3.3 The relationship between $f_i$ and $p_i$

We already defined the initial vacuum as in (3.1.15). By operating the annihilation operator $\boldsymbol{b}_i$ on the initial vacuum state $|0_-\rangle$, we obtain

$$\begin{aligned} \boldsymbol{b}_i|0_-\rangle &= \sum_j (\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger)|0_-\rangle \\ &= \sum_j -\beta_{ij}^* \boldsymbol{a}_j^\dagger |0_-\rangle \neq 0. \end{aligned} \tag{3.1.32}$$

Since $\beta_{ij} \neq 0$ in general, the initial vacuum state cannot be regarded as a vacuum state for an observer on $\mathscr{J}^+$. This means that particles are created.

We would like to find how many particles are created at $\mathscr{J}^+$ from the initial vacuum $|0_-\rangle$. By using the number operator $\boldsymbol{N}_i$ defined by

$$\boldsymbol{N}_i \equiv \boldsymbol{b}_i^\dagger \boldsymbol{b}_i, \tag{3.1.33}$$

we find that the vacuum expectation value of the particle number is given by

$$N_i \equiv \langle 0_-|\boldsymbol{N}_i|0_-\rangle \tag{3.1.34}$$

$$= \langle 0_-|\boldsymbol{b}_i^\dagger \boldsymbol{b}_i|0_-\rangle \tag{3.1.35}$$

$$= \sum_{j,k} \langle 0_-|\beta_{ik}\beta_{ij}^* \boldsymbol{a}_k \boldsymbol{a}_j^\dagger|0_-\rangle. \tag{3.1.36}$$

By using the commutation relation of the creation-annihilation operators given by

$$[\boldsymbol{a}_i, \boldsymbol{a}_j^\dagger] = \delta_{ij}, \tag{3.1.37}$$

the relation (3.1.36) becomes

$$N_i = \sum_{j,k} \beta_{ik}\beta^*_{ij}\delta_{jk} = \sum_j |\beta_{ij}|^2. \tag{3.1.38}$$

This stands for the number of particles which propagate to infinity among the particle pairs created by the vacuum. To determine the value, we calculate the coefficients $\beta_{ij}$.

By actually solving the Klein-Gordon equation (3.1.10), we obtain

$$f_{\omega' lm} = \frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}} e^{i\omega' v} Y_{lm}(\theta,\varphi), \tag{3.1.39}$$

$$p_{\omega lm} = \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega}} e^{i\omega u} Y_{lm}(\theta,\varphi). \tag{3.1.40}$$

For details of these calculations, see Appendix D. Here $Y_{lm}(\theta,\varphi)$ is the spherical harmonics, $l$ is an azimuthal quantum number, $m$ is a magnetic quantum number. The frequencies $\omega$ and $\omega'$ are eigenvalues given by

$$i\partial_t f_{\omega' lm} = \omega' f_{\omega' lm}, \tag{3.1.41}$$

$$i\partial_t p_{\omega lm} = \omega f_{\omega lm}. \tag{3.1.42}$$

Since the index of the state $i$ is uniquely determined by $\omega'$, $l$ and $m$, we represent $f_i$ as $f_{\omega' lm}$. The advanced time $v$ is an affine parameter at $\mathcal{J}^-$. The retarded time $u$ is an affine parameter at $\mathcal{J}^+$. They are defined as in (2.2.9). The solutions $f_{\omega' lm}$ and $p_{\omega lm}$ are obtained by approximating the Klein-Gordon equation at $r \to \infty$. Thus $F_{\omega'}(r)$ and $P_\omega(r)$ are integration constants containing a tiny effect depending on $r$.

By taking a continuous limit in (3.1.26), the relation (3.1.26) can be represented as

$$p_\omega = \int_0^\infty (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}) d\omega', \tag{3.1.43}$$

where we dropped indices $l$ and $m$ since the wave functions with different indices $l$ and $m$ are not connected to each other in a spherically symmetric system. In the continuous limit, the relations (3.1.30) and (3.1.38) become

$$\boldsymbol{b}_\omega = \int_0^\infty (\alpha_{\omega\omega'} \boldsymbol{a}_{\omega'} - \beta^*_{\omega\omega'} \boldsymbol{a}^\dagger_{\omega'}) d\omega', \tag{3.1.44}$$

$$N_\omega = \int_0^\infty |\beta_{\omega\omega'}|^2 d\omega'. \tag{3.1.45}$$

We can evaluate $\alpha_{ij}$ and $\beta_{ij}$ by performing the Fourier transform in (3.1.43). By substituting

(3.1.39) into (3.1.43) and multiplying the both sides by $\int_{-\infty}^{\infty} dv \exp(-i\omega'' v)$, we obtain

$$\int_{-\infty}^{\infty} dv e^{-i\omega'' v} p_\omega = 2\pi \int_0^{\infty} d\omega' \left[\alpha_{\omega\omega'} \frac{F_{\omega'}}{r\sqrt{2\pi\omega'}} \delta(\omega' - \omega'') + \beta_{\omega\omega'} \frac{F_{\omega'}}{r\sqrt{2\pi\omega'}} \delta(-\omega' - \omega'')\right]. \quad (3.1.46)$$

The second term on the right-hand side vanishes since $(\omega' + \omega'') \neq 0$. We thus obtain

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi} F_{\omega'}} \int_{-\infty}^{\infty} dv e^{-i\omega' v} p_\omega. \quad (3.1.47)$$

As for $\beta_{ij}$, we similarly obtain

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi} F_{\omega'}} \int_{-\infty}^{\infty} dv e^{i\omega' v} p_\omega. \quad (3.1.48)$$

Both (3.1.47) and (3.1.48) contain $u$ and $v$. We can derive the relation of between $u$ and $v$ from the following connection condition: We consider the wave function $p_\omega$ which reached $\mathcal{J}^+$. When we view it backwards, we can divide the wave function into two groups by how they propagate. The first group, which will be scattered by the Schwarzschild field outside the collapsing body, will propagate to $\mathcal{J}^-$. Then the wave function $p_\omega^{(1)}$ keeps the same frequency $\omega$ and propagate at $\mathcal{J}^-$. The second group will enter the collapsing body where it will be partly scattered and partly reflected through the center, eventually emerging to $\mathcal{J}^-$. It is this part $p_\omega^{(2)}$ which produces the interesting effect. Since the retarded time $u$ is infinite at the horizon, it is considered that the effective frequency of $p_\omega^{(2)}$ is enormous near the horizon. When the frequency is enormous, we can use the geometrical optics approximation. It means that the scattering of the wave function by the gravitational field can be ignored. We use the Penrose diagram in order to analyze the phase of $p_\omega^{(2)}$ (Fig. 3.4).

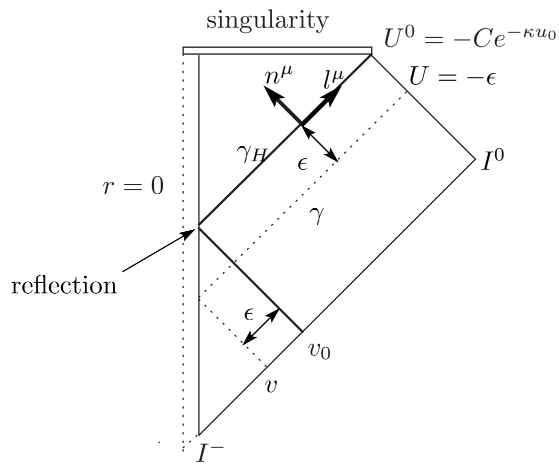


Fig. 3.4 Penrose diagram for the phase analysis of $p_i$

Although the Penrose diagram which contains the collapsing body is represented as in Fig. 2.7, we omit the collapsing body in Fig. 3.4 since we now consider the case that the frequency is enormous near the horizon. A coordinate $x$ is a point on the horizon outside the collapsing body, $l^\mu$ is a null vector which is tangent to the horizon at $x$ and $n^\mu$ is a null vector which is normal to the horizon at $x$ and is directed radially inwards. They are normalized so that

$$l^\mu n_\mu = -1. \tag{3.1.49}$$

The line $\mathcal{J}^+$ intersects the line $\mathcal{H}^+$ at a point. We represent the point by $u_0$ which is the affine parameter on $\mathcal{J}^+$. The $\gamma_{\rm H}$ is a null geodesic which goes back from $\mathcal{J}^+$ to backward. The null geodesic $\gamma_{\rm H}$ goes along the future event horizon $\mathcal{H}^+$, is reflected by following geometrical optics at $r = 0$ and reaches $\mathcal{J}^-$. We then represent the point on $\mathcal{J}^-$ by $v_0$ which is the affine parameter on $\mathcal{J}^-$. According to the Penrose diagram, an affine parameter $v$ on $\mathcal{J}^-$ is larger as it goes from $\mathcal{I}^-$ to $\mathcal{I}^0$. Thus, $v_0$ is the latest time that it leaves $\mathcal{J}^-$, passes through the center of the collapsing body and can escape to $\mathcal{J}^+$. Since the affine parameter $v$ which is larger than $v_0$ goes into the black hole through the horizon, it cannot escape to $\mathcal{J}^+$. For a very small constant $\epsilon$, a vector $-\epsilon n^\mu$ has a constant affine parameter $u$ on $\mathcal{J}^-$. From (3.1.40), the phase of the wave function $p_\omega^{(2)}$ is constant. We can also use geometrical optics approximation near the horizon because of a very small constant $\epsilon$. If null vectors $l^\mu$ and $n^\mu$ are translated along the null geodesic $\gamma_{\rm H}$ (the heavy line in Fig. 3.4), the phase of $p_\omega^{(2)}$ on the null geodesic $\gamma$ generated by $-\epsilon n^\mu$ remains constant in the

case that it enters inside the collapsing body. The null geodesic $\gamma$ is also reflected by geometrical optics at $r = 0$ and reaches $\mathcal{J}^-$.

The parameter $U$ is an affine parameter on the past event horizon $\mathcal{H}^-$. The parameter $U$ is such that at the point of intersection of the two horizon, $U = 0$ and $\dfrac{dx^\mu}{dU} = n^\mu$. The affine parameter $U$ is related to the retarded time $u$ on the past horizon by

$$U = -Ce^{-\kappa u}, \tag{3.1.50}$$

where $C$ is a constant and $\kappa$ is the surface gravity of the black hole defined by

$$\nabla_\nu K^\mu K^\nu = -\kappa K^\mu, \tag{3.1.51}$$

with $K^\mu$ the time translation Killing vector. By using this definition, we find that the surface gravity of a Schwarzschild black hole is given by

$$\kappa = \frac{1}{4M}. \tag{3.1.52}$$

The affine parameter $U$ is zero on the future horizon $\mathcal{H}^+$ and it satisfies $U = -\epsilon$ on the null geodesic $\gamma$ near the horizon. From (3.1.50), we obtain

$$u = -\frac{1}{\kappa}(\ln \epsilon - \ln C). \tag{3.1.53}$$

The phase of the wave function $p_\omega^{(2)}$ is connected to a point on $\mathcal{J}^-$ along $\gamma$. We represent the point by an affine parameter $v$. As found from Fig. 3.4, we obtain $\epsilon = v_0 - v$ on $\mathcal{J}^-$. Since the vector $n^\mu$ on $\mathcal{J}^-$ is parallel to the Killing vector $K^\mu$, the vector $n^\mu$ is given by

$$n^\mu = DK^\mu, \tag{3.1.54}$$

where $D$ is a constant. We thus find that $p_\omega^{(2)}$ is zero for $v > v_0$ because the particle is captured by the black hole and cannot escape to $\mathcal{J}^+$, while the phase of $p_\omega^{(2)}$ is given by (3.1.53) for $v > v_0$. The wave function then becomes

$$p_\omega^{(2)} \sim \begin{cases} 0, & (v > v_0), \\ \dfrac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i\dfrac{\omega}{\kappa}\ln\left(\dfrac{v_0 - v}{CD}\right)\right], & (v \le v_0), \end{cases} \tag{3.1.55}$$

where we used the fact that $v_0 - v$ is small and positive, and the definition $P_\omega^- \equiv P_\omega(2M)$. If we assume that $\omega'$ is very large, these would be determined by the asymptotic form

$$p_\omega^{(2)} \sim \frac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right]. \tag{3.1.56}$$

We can actually perform integrations of both (3.1.47) and (3.1.48). As a result, we obtain

$$\alpha^{(2)}_{\omega\omega'} \approx \frac{1}{2\pi} P^{-}_{\omega} (CD)^{\frac{i\omega}{\kappa}} e^{-i\omega' v_0} \left(\sqrt{\frac{\omega'}{\omega}}\right) \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (-i\omega')^{-1+\frac{i\omega}{\kappa}}, \tag{3.1.57}$$

$$\beta^{(2)}_{\omega\omega'} \approx -i\alpha^{(2)}_{\omega(-\omega')}. \tag{3.1.58}$$

For details of these calculations, see Appendix E. By expressing $\beta^{(2)}_{\omega\omega'}$ in terms of $\alpha^{(2)}_{\omega\omega'}$ from both (3.1.57) and (3.1.58), we obtain

$$\beta^{(2)}_{\omega\omega'} = e^{2i\omega' v_0} e^{[i\frac{\omega-1}{\kappa}]\ln(-1)} \alpha^{(2)}_{\omega\omega'}, \tag{3.1.59}$$

where we take $\ln(-1) = i\pi$ because we used the anticlockwise continuation around the singularity $\omega' = 0$. By substituting this relation into (3.1.59) and taking the absolute value, we obtain

$$|\beta^{(2)}_{\omega\omega'}| = e^{-\frac{\pi\omega}{\kappa}} |\alpha^{(2)}_{\omega\omega'}|, \tag{3.1.60}$$

where we note that this relation is valid for the large values of $\omega'$.

The expectation value of the total number of created particles at $\mathcal{J}^+$ in the frequency range $\omega$ to $\omega + d\omega$ is $d\omega \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2$. Since $|\beta_{\omega\omega'}|$ behaves as $(\omega')^{-\frac{1}{2}}$ by (3.1.58), this integral logarithmically diverges. It is considered that this infinite total number of created particles is caused since we evaluate a finite steady rate of emission for an infinite time. To evaluate the finite rate of emission, Hawking defined wave packets $p_{jn}$ by

$$p^{(2)}_{jn} \equiv \varepsilon^{-\frac{1}{2}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-\frac{2\pi i n\omega}{\varepsilon}} p^{(2)}_{\omega} d\omega, \tag{3.1.61}$$

where $j$ and $n$ are integers, $j \geq 0$, $\varepsilon \geq 0$. For small $\varepsilon$ these wave packets would have frequency $j\varepsilon$ and would be peaked around retarded time $u = \dfrac{2\pi n}{\varepsilon}$. We can expand $\{p_{jn}\}$ in terms of $\{f_\omega\}$

$$p^{(2)}_{jn} = \int_0^\infty (\alpha^{(2)}_{jn\omega'} f_{\omega'} + \beta^{(2)}_{jn\omega'} f^{*}_{\omega'}) d\omega'. \tag{3.1.62}$$

By comparing (3.1.62) with the relation (3.1.61) which is obtained by using (3.1.40), we find that the proportionality coefficient $\alpha_{jn\omega'}$ is defined by

$$\alpha^{(2)}_{jn\omega'} = \frac{1}{\sqrt{\varepsilon}} \int_{j\epsilon}^{(j+1)\varepsilon} e^{-2\pi i n\omega} \epsilon \alpha^{(2)}_{\omega\omega'} d\omega. \tag{3.1.63}$$

By substituting (3.1.57) into (3.1.63) for $j \gg \varepsilon$ and $n \gg \varepsilon$, we obtain

$$\begin{aligned} |\alpha^{(2)}_{jn\omega'}| &= \left| \frac{P^{-}_{\omega}}{2\pi\sqrt{\omega}} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) \frac{1}{\sqrt{\varepsilon\omega'}} \right| \int_{j\varepsilon}^{(j+1)\varepsilon} \exp\left[i\omega''\left(-\frac{2\pi n}{\varepsilon} + \frac{\log\omega'}{\kappa}\right) d\omega''\right] \\ &= \left| \frac{P^{-}_{\omega}}{\pi\sqrt{\omega}} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) \frac{\sin\frac{1}{2}\varepsilon z}{z\sqrt{\varepsilon\omega'}} \right|, \end{aligned} \tag{3.1.64}$$

where $\omega = j\epsilon$ and $z = \dfrac{1}{\kappa} = \ln \omega' - \dfrac{2\pi n}{\varepsilon}$. In these transformations, the relation (3.1.60) remains unchanged

$$|\beta^{(2)}_{jn\omega'}| = e^{-\frac{\pi\omega}{\kappa}} |\alpha^{(2)}_{jn\omega'}|. \tag{3.1.65}$$

Since the proportionality coefficient $|\beta_{jn\omega'}|$ thus behaves as $\sqrt{\dfrac{\varepsilon}{\omega'}}$, we can control the logarithmic divergence of the integral by an effect of $\varepsilon$. Therefore, the expectation value of the number of particles created and emitted to infinity $\mathcal{J}^-$ in the wave-packet mode $p_{jn}$, is given by

$$N_{jn} = \int_0^\infty |\beta^{(2)}_{jn\omega'}|^2 d\omega'. \tag{3.1.66}$$

We have considered the wave-packet $p_{jn}$ propagating backwards from $\mathcal{J}^+$. Until now, we have ignored the change in the amplitude of the wave function. However, a fraction of the particles would actually be scattered at the horizon. As a result, a fraction of the wave packet with $\rho(f_{jn}, f^*_{jn}) = 1$ as in (3.1.14) will be scattered by the static Schwarzschild field and the others will enter the collapsing body. Then the wave packets which reach $\mathcal{J}^+$ would satisfy $\rho(p_{jn}, p^*_{jn}) = \Gamma_{jn} < 1$ where $\Gamma_{jn}$ is called the gray body factor. The orthonormal condition (3.1.25) would become

$$\Gamma_{jn} = \int_0^\infty \left( |\alpha^{(2)}_{jn\omega'}|^2 - |\beta^{(2)}_{jn\omega'}|^2 \right) d\omega'. \tag{3.1.67}$$

By substituting (3.1.65) into (3.1.67), we obtain

$$\int_0^\infty |\beta^{(2)}_{jn\omega'}|^2 d\omega' = \frac{\Gamma_{jn}}{\exp(\frac{2\pi\omega}{\kappa}) - 1}. \tag{3.1.68}$$

From (3.1.66), the relation (3.1.68) is written as

$$N_{jn} = \frac{\Gamma_{jn}}{\exp(\frac{2\pi\omega}{\kappa}) - 1}. \tag{3.1.69}$$

This stands for the total number of particles created in the mode $p^{(2)}_{jn}$. If we ignore the gray body factor, the total number of particles $N$ is given by

$$N = \frac{1}{\exp(\frac{2\pi\omega}{\kappa}) - 1}. \tag{3.1.70}$$

In thermodynamics, the total number of particles for the black body radiation obeying Bose-Einstein statistics is given by

$$N = \frac{1}{\exp\left(\frac{\omega}{\mathcal{T}}\right) - 1}, \tag{3.1.71}$$

where $\omega$ is a frequency of the particle and $\mathcal{T}$ is temperature of the system. We thus find that a black hole which has Hawking temperature $\mathcal{T}_{\rm BH}$ defined by

$$\mathcal{T}_{\rm BH} = \frac{\kappa}{2\pi} \tag{3.1.72}$$

behaves as a black body and the black hole continuously emits radiation. Here $\kappa$ is the surface gravity of the black hole and we find that the temperature of the black hole is proportional to its surface gravity as already conjectured by the corresponding relationship between black hole physics and thermodynamics. We also find the black hole entropy $d\mathcal{S}_{\rm BH}$ from a thermodynamic consideration

$$d\mathcal{S}_{\rm BH} = \left(\frac{dM}{\mathcal{T}_{\rm BH}}\right). \tag{3.1.73}$$

By integrating Eq. (3.1.73), we obtain

$$\mathcal{S}_{\rm BH} = \frac{A}{4}, \tag{3.1.74}$$

where $A$ is the black hole area and we find that the black hole entropy is proportional to its area. Since it was shown that a black hole can radiate matter, we can regard that a black hole has temperature and entropy.

From the above result, Hawking suggested that a black hole can evaporate. The temperature of a Schwarzschild black hole is given by

$$\mathcal{T}_{\rm BH} = \frac{1}{8\pi M}, \tag{3.1.75}$$

where we used $\kappa = \dfrac{1}{4M}$ in (3.1.75). Namely the temperature of the black hole is inversely proportional to its mass. This means that the temperature is higher as the mass is smaller and the temperature is lower as the mass is larger (Tab. 3.1). It is known that the temperature for a black hole of the solar mass is much lower than the temperature of the cosmic microwave background radiation. Thus black holes of this size would be absorbing radiation faster than they emitted it and would be increasing its mass. However, there might be tiny black holes in the early universe [51, 84]. If the temperature of a tiny black hole is higher than the temperature of the cosmic microwave background radiation, such tiny black holes would be radiation-dominated. As this tiny black hole radiates matter, the mass becomes smaller, the temperature becomes higher and then it increasingly radiates matter. It would thus be expected that the black hole will evaporate at some point.

Tab. 3.1 The behavior of black hole

| | | | | |
|---|---|---|---|---|
| Large | $\Longleftarrow$ | Mass | $\Longrightarrow$ | Small |
| Low | $\Longleftarrow$ | Temperature | $\Longrightarrow$ | High |
| Absorption-dominated | $\Longleftarrow$ | Behavior of black hole | $\Longrightarrow$ | Radiation-dominant |

Hawking radiation can also be shown for not only a Schwarzschild black hole but also for other black holes. In the case of a Kerr-Newman black hole, the relation (3.1.70) is extended to

$$N = \frac{1}{\exp\left[\frac{2\pi}{\kappa}\left(\omega - m\Omega_{\mathrm{H}} - e\Phi_{\mathrm{H}}\right)\right] - 1}, \tag{3.1.76}$$

where $m$ is a magnetic quantum number of the emitted matter field, $e$ is the charge of the matter field, $\Omega_{\mathrm{H}}$ is the angular velocity of the black hole, $\Phi_{\mathrm{H}}$ is the electrical potential of the black hole and $\kappa$ is given by not $\dfrac{1}{4M}$ but by $\dfrac{4\pi(r_+ - M)}{A}$ as in (2.1.23).

Since the black holes actually have temperature and entropy, the first law of the black hole physics is written as

$$dM = \mathcal{T}_{\mathrm{BH}} d\mathcal{S}_{\mathrm{BH}} + \Omega_{\mathrm{H}} dL + \Phi_{\mathrm{H}} dQ, \tag{3.1.77}$$

and the second law is given by

$$\Delta\mathcal{S}_{\mathrm{BH}} + \Delta\mathcal{S}_{\mathrm{C}} = \Delta(\mathcal{S}_{\mathrm{BH}} + \mathcal{S}_{\mathrm{C}}) \geq 0, \tag{3.1.78}$$

where $\mathcal{S}_{\mathrm{C}}$ is the entropy of the matter outside the black hole. It was shown that black holes can radiate by using quantum effects. As was shown in Section 2.3, although a part of energy can be extracted from a rotating black hole by the Penrose process, this cannot break the classical Hawking's black hole area theorem (2.5.2). On the other hand, Hawking radiation decreases its black hole area, and the classical Hawking's black hole area theorem is violated [54]. Thus one needs to generalize the second law as in (3.1.78). This consideration was already performed by Bekenstein [38, 53] before Hawking's original paper [2]. The generalized second law always holds in any physical process.

## 3.2 Previous Works on Hawking Radiation

After Hawking's original derivation, various derivations of Hawking radiation have been suggested. In this section, we would like to review some representative derivations of Hawking radiation

very briefly. For sake of simplicity, we consider the case of a Schwarzschild black hole unless stated otherwise.

Firstly, we review the derivation by using the path integral [55], which was suggested by Gibbons and Hawking. The basic 4-dimensional action for the gravitational field is commonly given by the Einstein-Hilbert action,

$$S = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{\pm h}K, \tag{3.2.1}$$

where $R$ is the scalar curvature, $K$ is the trace of extrinsic curvature $K_{\mu\nu}$, $\partial\mathcal{M}$ is a suitable boundary of a manifold $\mathcal{M}$ and $h$ is the determinant of $h_{\mu\nu}$ which is the induced metric of the boundary $\partial\mathcal{M}$. Of course, we can derive the Einstein equation (2.1.1) by considering the variation of $g_{\mu\nu}$ in (3.2.1).

Since we consider the case of a Schwarzschild background, the metric is given by the Schwarzschild metric,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2d\Omega^2. \tag{3.2.2}$$

It is well known that the metric has singularities both at the origin and the horizon. To remove a fictitious singularity at the horizon, the Kruskal-Szekeres coordinates are often used,

$$ds^2 = \frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)(-dz^2+dy^2) + r^2d\Omega^2, \tag{3.2.3}$$

where

$$-z^2+y^2 = \left(\frac{r}{2M}-1\right)\exp\left(\frac{r}{2M}\right), \tag{3.2.4}$$

$$\frac{y+z}{y-z} = \exp\left(\frac{t}{2M}\right). \tag{3.2.5}$$

Here we define the imaginary time by

$$\tau \equiv it. \tag{3.2.6}$$

From (3.2.5), we then find that $\tau$ is periodic with the period of $8\pi M$. By substituting the Euclidean metric associated with (3.2.2) into the Euclidean action associated with (3.2.1), we can evaluate the action integral.

According to quantum field theory, in the path integral approach to the quantization of a real scalar field $\phi$, we can represent the transition amplitude to go from $\phi_1$ at a time $t_1$ to $\phi_2$ at a time

$t_2$ as

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int \mathcal{D}\phi e^{iS(\phi)}, \tag{3.2.7}$$

where the path integral is over all field configurations. On the other hand, in the operator formulation, the transition amplitude is given by

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2 | e^{-iH(t_2 - t_1)} | \phi_1 \rangle, \tag{3.2.8}$$

where $H$ is the Hamiltonian. We define a Euclidean time with a certain period as $t_2 - t_1 = -i\beta$ and we then set $\phi_1 = \phi_2$. By taking the sums over all $\phi_1$, we obtain

$$\sum_{\phi_1} \langle \phi_1 | e^{-\beta H} | \phi_1 \rangle = \mathrm{Tr}\left(e^{-\beta H}\right). \tag{3.2.9}$$

According to quantum statistical mechanics, the right-hand side of this relation just corresponds to the partition function $Z$ for the canonical ensemble consisting of the field $\phi$ at temperature $T = \dfrac{1}{\beta}$, i.e.,

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right). \tag{3.2.10}$$

We thus find that the partition function of the system is represented as the path integral with a periodic Euclidean time. According to statistical mechanics, it is known that the entropy is represented in terms of the partition function as

$$\mathcal{S} = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z. \tag{3.2.11}$$

In the black hole background, we found that the system has the period by taking the Euclidean time in (3.2.5). Furthermore, the Euclidean path integral presents the partition function and we can then derive the entropy by using the approximative treatment both of the path integral and Smarr's formula [56]. This agrees with Hawking's original result. In this sense, Gibbons and Hawking's derivation is very simple and provides universal picture of the black hole entropy. However, on the other hand, it reveals mysterious results that the black hole entropy is generated simply by a Legendre transformation from Hamiltonian picture to Lagrangian picture in the path integral.

In connection with this derivation, the approach which uses both the Legendre transformation and the consideration based on the change of the topology, has been analyzed by Bañados, Teitelboim and Zanelli [57], and also by Hawking and Horowitz [58]. In particular, the derivation of

Bañados, Teitelboim and Zanelli using the Gauss-Bonnet theorem, is interesting. The Euclidean Einstein-Hilbert action is written as

$$S_{\mathrm{E}} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{g}g^{\mu\nu}R^{\alpha}_{\mu\alpha\nu} + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K, \tag{3.2.12}$$

while the Gauss-Bonnet theorem for a 2-dimensional manifold with a suitable boundary is written as

$$\frac{1}{2}\int_{\mathcal{M}} dx^2\sqrt{g}g^{\mu\nu}R^{\alpha}_{\mu\alpha\nu} + \int_{\partial\mathcal{M}} dx\sqrt{h}K = 2\pi\chi(\mathcal{M}), \tag{3.2.13}$$

where $\chi(\mathcal{M})$ is the Euler number of $\mathcal{M}$ which depends solely on its topology. For example, $\chi(\mathcal{M}) = 1$ for a disk and $\chi(\mathcal{M}) = 0$ for an annulus. By using the Gauss-Bonnet theorem, the important role of topologies becomes clear. The relationship between the action and the entropy in the above arguments is also related by the Legendre transformation. Although this result agrees with Hawking's original one, it has not been presented in the past as an explicit manner as the basic reason why the entropy is generated by the Legendre transformation and the change of topologies.

Secondly, the derivation of Hawking radiation from the calculation of the energy-momentum tensor in a black hole background was suggested by Christensen and Fulling [59]. They determined the form of the energy momentum tensor by using symmetry arguments and the conservation law of the energy-momentum tensor by taking into consideration of the trace anomaly which is given by

$$T^{\alpha}_{\ \alpha} = \frac{1}{24\pi}R, \tag{3.2.14}$$

where $R$ is the scalar curvature in a 2-dimensional theory (for example, see [60]). This anomaly appears as a quantum contribution to the trace $T^{\alpha}_{\ \alpha}$ of the energy-momentum tensor. By requiring that the energy-momentum tensor is finite as seen by a free falling observer at the horizon in a 2-dimensional Schwarzschild background and imposing the anomalous trace equation everywhere, we can obtain the characteristic flux

$$F_{\mathrm{H}} = \frac{M}{2}\int_{2M}^{\infty}\frac{dr}{r^2}T^{\alpha}_{\ \alpha}(r), \tag{3.2.15}$$

which corresponds to the Hawking's result. However, this result is valid for a 2-dimensional theory only. In a 4-dimensional theory, there remains an indeterminable function and the all energy-momentum tensor cannot be determined by symmetries alone. Therefore, this derivation has the weakness that it is not valid for a 4-dimensional theory.

Thirdly, we would like to refer to the derivation of the black hole entropy based on the idea of entanglement entropy. The entanglement entropy is understood as a measure of the information loss due to a division of the system. We consider that the total system can be divided into two subsystems. The Hilbert space of the total system $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \tag{3.2.16}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ stand for the two subsystems and $\otimes$ is the tensor product. Roughly speaking, if a state cannot be written as a product of the states in each subspace, namely, the state is not separable, it is called the entangled state. (Recently, the criterion of entanglement was quantitatively clarified by Fujikawa [61, 62]. ) The entanglement entropy is quantitatively defined by the quantum von Neumann entropy

$$\mathcal{S}_{12} = -\mathrm{Tr}_1(\rho_{\mathrm{red}} \ln \rho_{\mathrm{red}}), \tag{3.2.17}$$

where $\rho_{\mathrm{red}}$ is the reduced density matrix to the space $\mathcal{H}_1$ and the trace is taken over the states of $\mathcal{H}_1$. This is understood as a generalization of usual entropy in thermodynamics. We also note that one of the important properties of the entanglement entropy is that it is symmetric under an interchange of the role of $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$S_{12} = S_{21}. \tag{3.2.18}$$

Even if the sizes of two subspaces are different as drawn in Fig 3.5, it means that the two entropies agree with each other. We find that the entanglement entropy possesses properties differing from the usual entropy which is an extensive quantity in statistical mechanics. For details, see [63, 64].

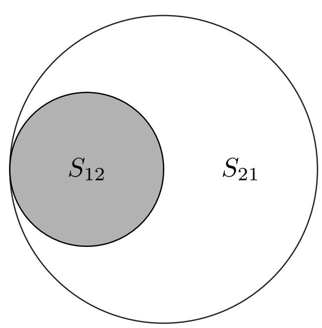


Fig. 3.5 The symmetry of the entanglement entropy

The derivation of the black hole entropy based on the entanglement entropy was analyzed by Terashima [65]. When the concept of the entanglement entropy is applied to the case of a black hole, we can understand the black hole entropy as the information loss due to a spatial separation by the appearance of the horizon. He regarded the outside and a thin region (of the order of the Planck length) inside of the horizon as each subsystem and evaluated the entanglement entropy. In this derivation, the origin of the black hole entropy is very clear, while the coefficient of the black hole entropy cannot be exactly determined.

Fourthly, although it is not the case of a Schwarzschild black hole, we would like to state the derivation of the black hole entropy based on the consideration of string theory [66, 67]. This derivation was suggested by Strominger and Vafa [66]. In the 5-dimensional extremal black hole made by "D-branes", they directly evaluated the entropy by counting the number of the BPS states which partly preserve supersymmetry. In this derivation, the origin of the black hole entropy is very clear, and furthermore this result surprisingly agrees with the previous result including the numerical coefficient of the black hole entropy. However, this derivation is valid for quite atypical black holes only. The method has not been extended yet to the case of a well-known black hole with a finite temperature such as a Schwarzschild black hole.

Finally, from the above discussions, we find that there are strengths and weaknesses in each derivation. In the method directly deriving the black hole entropy from the point of view of statistical mechanics or information theory [38, 65], we cannot determine the correct coefficient of the black hole entropy for black holes with a finite temperature. The derivation by string theory [66], where the origin of the entropy is clear and furthermore the result is correctly reproduced up to the coefficient of the black hole entropy, is valid for black holes with zero temperature (extremal) which means that the black hole does not exhibit any radiation. In the methods which derive the temperature of the black hole [2, 3, 7, 55, 59], although approximation procedures are used in each derivation, the correct result including the coefficient of the temperature is derived. We may say that the origin of the black hole entropy is not sufficiently understood yet. As recent attempts toward the better understanding of black hole radiation, we will discuss the derivation of Hawking radiation from the tunneling mechanism [7] in chapter 5 and from anomalies [3] in chapter 4, respectively.

# Chapter 4

# Hawking Radiation and Anomalies

Robinson and Wilczek suggested a new method of deriving Hawking radiation by the consideration of anomalies [3]. The basic idea of their approach is that the flux of Hawking radiation is determined by anomaly cancellation conditions in the Schwarzschild black hole background. Iso, Umetsu and Wilczek, and also Murata and Soda, extended the method to a charged Reissner-Nordström black hole [4] and a rotating Kerr black hole [5, 68], and they showed that the flux of Hawking radiation can also be determined by anomaly cancellation conditions and regularity conditions of currents at the horizon. Their formulation thus gives the correct Hawking flux for all the cases at infinity and thus provides a new attractive method of understanding Hawking radiation. We present some arguments which clarify this derivation [6]. We show that the Ward identities and boundary conditions for covariant currents, without referring to the Wess-Zumino terms and the effective action, are sufficient to derive Hawking radiation. Our method, which does not use step functions, thus simplifies some of the technical aspects of the original formulation described above.

The contents of this chapter are as follows. In Section 4.1, we briefly review quantum anomalies in quantum field theory. In Section 4.2, we would like to discuss the connection between Hawking radiation and quantum anomalies. In Section 4.3, we clarify some arguments in previous works with respect to the derivation of Hawking radiation from anomalies and present a simplified derivation.

## 4.1 Quantum Anomaly

Quantum anomaly is one of important phenomena in quantum field theory. This phenomenon is closely related to the concept of symmetry in field theory. The symmetry plays a very important

role in modern physics. It is well known that if a system has symmetries, there are corresponding conserved quantities which are dictated by the Nöther's theorem [69]. For example, if a system has the time translation symmetry, the corresponding conserved quantity is the energy of the system and the energy conservation law holds. Also if a system has the rotational symmetry, the angular momentum is conserved and the angular momentum conservation law holds.

From this viewpoint, a quantum anomaly represents the fact that Nöther's theorem can be broken by quantization. Even if there is a certain symmetry and the corresponding conservation law exists in a classical theory, it is possible that the symmetry is broken in a quantized theory. This symmetry breaking is commonly called "quantum anomaly" or simply "anomaly".

Historically the quantum anomaly was first discovered in the evaluation of the two-photon decay of the neutral $\pi$ meson by Fukuda and Miyamoto [70]. Afterward, it was clarified that the anomaly is an inevitable phenomenon in the local field theory by Bell and Jackiw [71], and Adler [72]. In the path integral formulation, it was shown that an anomaly is formulated by a Jacobian in the change of path integral variables by Fujikawa [73].

The chiral symmetry is known as a famous symmetry which causes the anomaly. This symmetry is a relatively-new symmetry which was discovered by the introduction of the Dirac equation. In this section, we would like to discuss the chiral anomaly as an example of anomalies in 4-dimensional Minkowski space-time.

Nöther's theorem and anomalies can be simply described in the path integral formulation. The pass integral for a fermion field $\psi(x)$ is defined by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{iS\}, \tag{4.1.1}$$

where $S$ is the action of the system. The path integral measure is defined by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \equiv \prod_x \frac{\delta}{\delta\bar{\psi}(x)} \prod_y \frac{\delta}{\delta\psi(y)}, \tag{4.1.2}$$

and the fermion field $\psi(x)$ is also called the Dirac field which satisfies the anticommutation relation (Grassmann number) in the classical level

$$\{\psi(x), \psi(y)\} = \left\{\psi(x), \bar{\psi}(y)\right\} = \left\{\bar{\psi}(x), \bar{\psi}(y)\right\} = 0, \tag{4.1.3}$$

and the Dirac conjugate $\bar{\psi}(x)$ is defined by

$$\bar{\psi}(x) \equiv \psi^{\dagger}(x)\gamma^0. \tag{4.1.4}$$

In quantum electrodynamics, the action for a massless Dirac field is given by

$$S(\psi, \bar{\psi}, V_\mu) = \int d^4x \bar{\psi}(x) \left[ i\gamma^\mu \left( \partial_\mu - ieV_\mu(x) \right) \right] \psi(x), \tag{4.1.5}$$

where $V_\mu(x)$ is the gauge field, $e$ is the charge of the fermion field and $\gamma^\mu$ is called the gamma matrix or the Dirac matrix defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{4.1.6}$$

$$(\gamma^0)^\dagger = \gamma^0, \tag{4.1.7}$$

$$(\gamma^k)^\dagger = -\gamma^k, \qquad (k = 1, 2, 3), \tag{4.1.8}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. By substituting (4.1.5) into (4.1.1), we obtain the path integral of quantum electrodynamics

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ i \int d^4x \bar{\psi}(x) \left[ i\gamma^\mu \left( \partial_\mu - ieV_\mu(x) \right) \right] \psi(x) \right\}. \tag{4.1.9}$$

First we consider the local $U(1)$ gauge transformation defined by

$$\psi'(x) = e^{i\alpha(x)}\psi(x), \tag{4.1.10}$$

$$\bar{\psi}'(x) = \bar{\psi}(x)e^{-i\alpha(x)}, \tag{4.1.11}$$

$$V'_\mu(x) = V_\mu + \frac{1}{e}\partial_\mu\alpha(x). \tag{4.1.12}$$

We find that the action (4.1.5) is invariant under this transformation

$$S(\psi', \bar{\psi}', V'_\mu) = S(\psi, \bar{\psi}, V_\mu). \tag{4.1.13}$$

By the fact that the value of a definite integral does not depend on the naming of integration variables, we have the identity

$$\int \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \exp\left\{ iS(\psi', \bar{\psi}', V'_\mu) \right\} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ iS(\psi, \bar{\psi}, V'_\mu) \right\}. \tag{4.1.14}$$

By substituting (4.1.13) into (4.1.14), we obtain

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ iS(\psi, \bar{\psi}, V_\mu) \right\} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{ iS(\psi, \bar{\psi}, V'_\mu) \right\}, \tag{4.1.15}$$

where we assumed that the integral measure is invariant under the above gauge transformation

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi. \tag{4.1.16}$$

The relation (4.1.15) is called the Ward identity. By substituting (4.1.12) into (4.1.5) and integrating by parts, we obtain

$$S(\psi, \bar{\psi}, V'_\mu) = \int d^4x \bar{\psi}(x) \left[ i\gamma^\mu \left( \partial_\mu - ieV_\mu(x) \right) \right] \psi(x) + \int d^4x \alpha(x) \partial_\mu (\bar{\psi}(x) \gamma^\mu \psi(x)). \qquad (4.1.17)$$

The Ward identity (4.1.15) thus becomes

$$i \int d^4x \alpha(x) \partial_\mu \langle (\bar{\psi}(x) \gamma^\mu \psi(x)) \rangle = 0, \qquad (4.1.18)$$

where we chose $\alpha(x)$ as an infinitely small parameter and we defined the expectation value of an operator $\hat{O}(x)$ by

$$\langle O(x) \rangle \equiv \int \mathcal{D}\bar{\psi} \mathcal{D}\psi O(x) \exp\left[ iS \right]. \qquad (4.1.19)$$

From the identity (4.1.18), we can obtain the current conservation law

$$\partial_\mu J^\mu(x) = 0, \qquad (4.1.20)$$

where $J^\mu(x)$ is the Nöther current defined by

$$J^\mu(x) \equiv \langle (\bar{\psi}(x) \gamma^\mu \psi(x)) \rangle, \qquad (4.1.21)$$

which stands for the quantized quantity in the operator formalism. From the above discussion, Nöther's theorem corresponds to the fact that the integral measure is invariant under the transformation of integration variables in the path integral formulation.

Next we consider the chiral transformation

$$\psi'(x) = e^{i\gamma_5 \alpha(x)} \psi(x), \qquad (4.1.22)$$

$$\bar{\psi}'(x) = \bar{\psi}(x) e^{i\gamma_5 \alpha(x)}, \qquad (4.1.23)$$

where $\gamma_5$ is defined by

$$\gamma_5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \qquad (4.1.24)$$

$$\{\gamma_5, \gamma^\mu\} = 0. \qquad (4.1.25)$$

It is shown that the action becomes

$$S(\psi', \bar{\psi}', V_\mu) = S(\psi, \bar{\psi}, V_\mu) + \int d^4x \alpha(x) \partial_\mu (\bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x)) \qquad (4.1.26)$$

under the chiral transformation and the integral measure is transformed as [75]

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[-i\int d^4x\alpha(x)\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\right], \tag{4.1.27}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita symbol and $F_{\mu\nu}$ is the field strength tensor defined by

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu. \tag{4.1.28}$$

For details of this calculation, see, for example, §5.1 in [75]. By using the Ward identity, it is shown that the chiral anomaly is given by

$$\partial_\mu J_5^\mu(x) = \frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}, \tag{4.1.29}$$

where $J_5^\mu(x)$ stands for the chiral current defined by

$$J_5^\mu(x) \equiv \langle\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\rangle. \tag{4.1.30}$$

According to classical theory, it can be shown that both the gauge current $J^\mu(x)$ and the chiral current $J_5^\mu(x)$ are conserved. By the quantization procedure, the gauge current is conserved as in (4.1.20) and the gauge symmetry thus holds, while the chiral current is not conserved as in (4.1.29) and the chiral symmetry is broken. As found from the form of (4.1.29), the current conservation law is broken and we can therefore regard the "anomaly" as "generating a source of the current" by quantization. This picture is useful to understand the following discussions.

## 4.2 Derivation of Hawking Radiation from Anomalies

Robinson and Wilczek demonstrated a new method of deriving Hawking radiation [3]. They derived Hawking radiation by the consideration of quantum anomalies. Their derivation has an important advantage in localizing the source of Hawking radiation near the horizon where anomalies are visible. Since both of the anomalies and Hawking radiation are typical quantum effects, it is natural that Hawking radiation is related to anomalies. Iso, Umetsu and Wilczek improved the approach of [3] and extended the method to a charged Reissner-Nordström black hole [4]. This approach was also extended to a rotating Kerr black hole and a charged and rotating Kerr-Newman black hole by Murata and Soda [68] and by Iso, Umetsu and Wilczek [5].

The essential idea of Iso, Umetsu and Wilczek [5] is the following. They consider a quantum field in a black hole background. As shown in Section 2.4, by using the technique of the dimensional reduction, the field can be effectively described by an infinite collection of $(1 + 1)$-dimensional

fields on $(t,r)$ space near the horizon. Then the mass or potential terms of quantum fields can be suppressed near the horizon. Therefore we can treat the 4-dimensional theories as a collection of 2-dimensional quantum fields. In this 2-dimensions, outgoing modes near the horizon behave as right moving modes while ingoing modes as left moving modes. Since the horizon is a null hypersurface, all ingoing modes at the horizon can not classically affect physics outside the horizon (see Fig. 4.1, where we utilized the Penrose diagram in the Schwarzschild background for simplicity). Then, if we integrate the ingoing modes to obtain the effective action in the exterior region, it becomes anomalous with respect to gauge or general coordinate symmetries since the effective theory is now chiral at the horizon. The underlying theory is of course invariant under these symmetries and these anomalies must be cancelled by quantum effects of the classically irrelevant ingoing modes. They showed that the condition for anomaly cancellation at the horizon determines the Hawking flux of the charge and energy-momentum. The flux is universally determined only by the value of anomalies at the horizon.

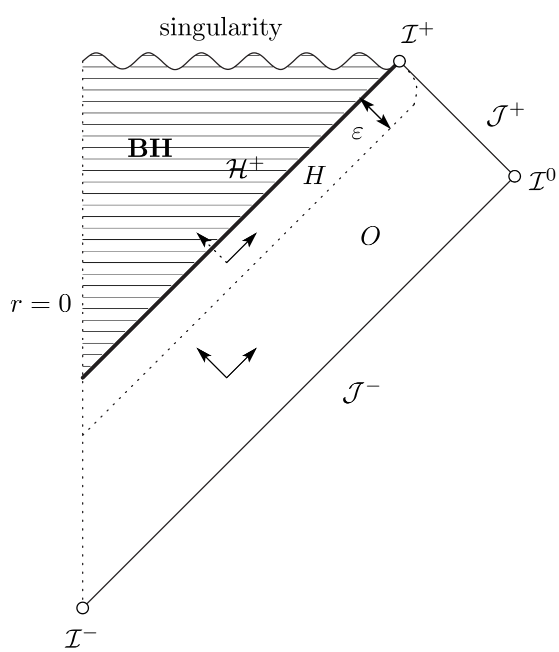


Fig. 4.1 The Penrose diagram relevant to the analysis of Robinson and Wilczek

The approach of Iso, Umetsu and Wilczek [5] is very transparent and interesting. However, there remain several points to be clarified. First, Iso, Umetsu and Wilczek start by using both

the consistent and covariant currents. However, they only impose boundary conditions on covariant currents. As discussed in [4], it is not clear why we should use covariant currents instead of consistent ones to specify the boundary conditions at the horizon. Banerjee and Kulkarni considered an approach using only covariant currents without consistent currents [76]. However, their approach heavily relies on the Wess-Zumino terms defined by consistent currents [77]. The Wess-Zumino terms are also used in the approach of Iso, Umetsu and Wilczek. Therefore, Banerjee and Kulkarni's approach is not completely described by covariant currents only.

Second, in the approach of Iso, Umetsu and Wilczek the region outside the horizon must be divided into two regions because the effective theories are different near and far from the horizon. They thus used step functions to divide these two regions. We think that the region near the horizon and the region far from the horizon are continuously related. Nevertheless, if one uses step functions, terms with delta functions, which originate from the derivatives of step functions when one considers the variation of the effective action, appear. They disregarded the extra terms by claiming that these terms correspond to the contributions of the ingoing modes. This is the second issue that we wish to address here. Banerjee and Kulkarni also considered an approach without step functions [78]. They obtained the Hawking flux by using the effective actions and two boundary conditions for covariant currents. However, they assumed that the effective actions are 2-dimensional in both the region near the horizon and the region far from the horizon [79, 80]. As already discussed in the approach of Iso, Umetsu and Wilczek, the original 4-dimensional theory is the 2-dimensional effective theory in the region near the horizon. However, the effective theory should be 4-dimensional in the region far from the horizon.

In contrast with the above approaches, we derive the Hawking flux using only the Ward identities and two boundary conditions for the covariant currents. We formally perform the path integral, and the Nöther currents are constructed by the variational principle. Therefore, we can naturally treat the covariant currents [73, 74, 81]. We do not use the Wess-Zumino term, the effective action or step functions. Therefore, we do not need to define consistent currents. Although we use the two boundary conditions used in Banerjee and Kulkarni's method, we use the 4-dimensional effective theory far from the horizon and the 2-dimensional theory near the horizon. In this sense, our method corresponds to the method of Iso, Umetsu and Wilczek. It is easier to understand the derivation of the Ward identities directly from the variation of matter fields than their derivation from the effective action since we consider Hawking radiation as resulting from the effects of matter fields.

Our approach is essentially based on the approach of Iso, Umetsu and Wilczek. However, we simplify the derivation of Hawking radiation by clarifying the above issues. We only use the Ward identities and two boundary conditions for covariant currents, and we do not use the Wess-Zumino terms, the effective action or step functions, as stated above. In the next section, we will show our simple derivation.

## 4.3 Ward Identity in the Derivation of Hawking Radiation from Anomalies

In this section, we would like to clarify some arguments in previous works and present a simple derivation of Hawking radiation from anomalies. By using the Ward identities and two boundary conditions only, we show how to derive the Hawking flux.

The contents of this section are as follows. In Subsection 4.3.1, we show a simple derivation of Hawking radiation from anomalies for the case of a Kerr black hole. In Subsection 4.3.2, we discuss differences among our work and previous works. In Subsection 4.3.3, we also show that the Hawking flux can be derived for the case of a Reissner-Nordström black hole by using our approach.

### 4.3.1 The case of a Kerr black hole

To compare our method with the approach of Iso, Umetsu and Wilczek [5], we consider the Kerr black hole background. By taking $Q = 0$ in the Kerr-Newman metric which is given by (2.1.8), we obtain the Kerr metric

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - \frac{2a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta) dt d\varphi \\ &- \frac{a^2 \Delta \sin^2\theta - (r^2+a^2)^2}{\Sigma} \sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned} \tag{4.3.1}$$

with

$$\Sigma \equiv r^2 + a^2 \cos^2\theta, \tag{4.3.2}$$

$$\Delta \equiv r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \tag{4.3.3}$$

where $r_{+(-)}$ is the radius of the outer (inner) horizon

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{4.3.4}$$

The action for a scalar field is given by

$$S_{(O)} = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + S_{\rm int} \tag{4.3.5}$$

as in (2.4.1). We note that no gauge field exists in (4.3.5). By using the technique of the dimensional reduction as shown in Section 2.4, we obtain the effective (1+1)-dimensional action near the horizon

$$S_{(H)} = -\sum_{l,m}\int dtdr\Phi\phi^*_{lm}\left[g^{tt}\left(\partial_t - imU_t\right)^2 + \partial_r g^{rr}\partial_r\right]\phi_{lm}, \tag{4.3.6}$$

with

$$\Phi = r^2 + a^2, \tag{4.3.7}$$
$$g_{tt} = -f(r), \quad g_{rr} = \frac{1}{f(r)}, \quad g_{rt} = 0, \tag{4.3.8}$$
$$f(r) \equiv \frac{\Delta}{r^2+a^2}, \tag{4.3.9}$$
$$U_t = -\frac{a}{r^2+a^2},\ U_r = 0, \tag{4.3.10}$$

where $\Phi$ is the dilaton field, $U_\mu$ is the $U(1)$ gauge field and $m$ is the $U(1)$ charge.

From (4.3.6), we find that the effective theory is the (1+1)-dimensional theory near the horizon. However, we cannot simply regard the effective theory far from the horizon as $(1+1)$-dimensional theory. We need to divide the region outside the horizon into two regions because the effective theories are different near the horizon and far from the horizon. We define region $O$ as the region far from the horizon and region $H$ as the region near the horizon. Note that the action in region $O$ is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}]$ and the action in region $H$ is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, U_\mu, \Phi]$.

We can divide the field associated with a particle into ingoing modes falling toward the horizon (left-handed) and outgoing modes moving away from the horizon (right-handed) using a Penrose diagram [3–5] (Fig. 4.2). Since the horizon is a null hypersurface, none of the ingoing modes at the horizon are expected to affect the classical physics outside the horizon. Thus, we ignore the ingoing modes. Therefore, anomalies appear with respect to the gauge or general coordinate symmetries since the effective theory is chiral near the horizon. Here, we do not consider the backscattering of ingoing modes, i.e., the gray body radiation.

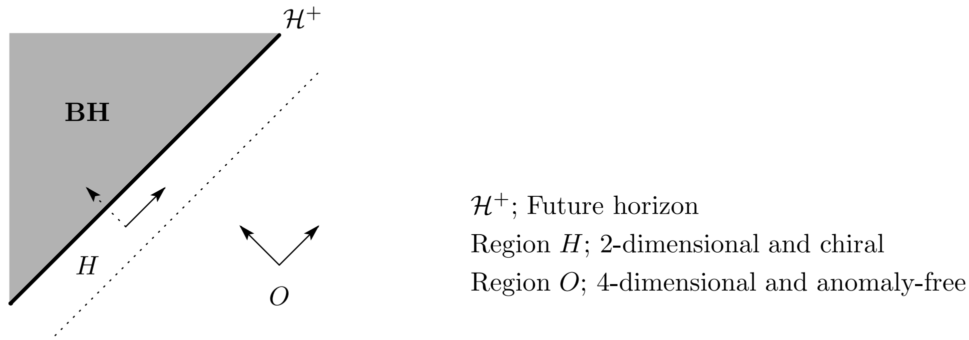


Fig. 4.2 Part of the Penrose diagram relevant to our analysis. The dashed arrow in region $H$ represents the ignored ingoing mode falling toward the horizon.

We now present the derivation of Hawking radiation for the Kerr black hole. First, we consider the effective theory in region $O$. The effective theory is 4-dimensional in region $O$, which we cannot reduce to a 2-dimensional theory. In contrast to the case of a charged black hole, a 4-dimensional gauge field such as the Coulomb potential $V_t$ does not exist in a rotating Kerr black hole. Therefore, we do not define the $U(1)$ gauge current in region $O$. On the other hand, the effective theory in region $H$ is a 2-dimensional chiral theory and we can regard a part of the metric as a gauge field such as (4.3.10), since the action of (4.3.6) is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, U_\mu, \Phi]$.

Second, we consider the Ward identity for the gauge transformation in region $H$ near the horizon. Here, we pretend to formally perform the path integral for $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, U_\mu, \Phi]$, where the Nöther current is constructed by the variational principle, although we do not perform an actual path integral. Therefore, we can naturally treat *covariant* currents [73]. As a result, we obtain the Ward identity with a gauge anomaly

$$\nabla_\mu J^\mu_{(H)} - \mathscr{C} = 0, \tag{4.3.11}$$

where we define covariant currents $J^\mu_{(H)}(r)$ and $\mathscr{C}$ is a covariant gauge anomaly. This Ward identity is for right-handed fields. The covariant form of the 2-dimensional Abelian anomaly $\mathscr{C}$ is given by

$$\mathscr{C} = \pm \frac{m^2}{4\pi\sqrt{-g_{(2)}}} \epsilon^{\mu\nu} F_{\mu\nu}, \quad (\mu, \nu = t, r) \tag{4.3.12}$$

where $+(-)$ corresponds to right(left)-handed matter fields, $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{tr} = 1$ and $F_{\mu\nu}$ is the field-strength tensor. Using the 2-dimensional metric (4.3.8), the identity

(4.3.11) is written as

$$\partial_r J^r_{(H)}(r) = \frac{m^2}{2\pi}\partial_r U_t(r). \tag{4.3.13}$$

By integrating Eq. (4.3.13) over $r$ from $r_+$ to $r$, we obtain

$$J^r_{(H)}(r) = \frac{m^2}{2\pi}\left[U_t(r) - U_t(r_+)\right], \tag{4.3.14}$$

where we use the condition

$$J^r_{(H)}(r_+) = 0. \tag{4.3.15}$$

The condition (4.3.15) corresponds to the statement that free falling observers see a finite amount of the charged current at the horizon, i.e., (4.3.15) is derived from the regularity of covariant currents. This condition was used in the approach of Iso, Umetsu and Wilczek [5]. We regard (4.3.14) as a covariant $U(1)$ gauge current appearing in region $H$ near the horizon.

Third, we consider the Ward identity for the general coordinate transformation in region $O$ far from the horizon. By improving the approach of [5], we define the formal 2-dimensional energy-momentum tensor in region $O$ from the exact 4-dimensional energy-momentum tensor in region $O$ and we connect the 2-dimensional energy-momentum tensor thus-defined in region $O$ with the 2-dimensional energy-momentum tensor in region $H$. Since the action is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}]$ in region $O$, the Ward identity for the general coordinate transformation is written as

$$\nabla_\nu T^{\mu\nu}_{(4)} = 0, \tag{4.3.16}$$

where $T^{\mu\nu}_{(4)}$ is the 4-dimensional energy-momentum tensor. Since the Kerr background is stationary and axisymmetric, the expectation value of the energy-momentum tensor in the background depends only on $r$ and $\theta$, i.e., $\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}(r,\theta)\rangle$. The $\mu = t$ component of the conservation law (4.3.16) is written as

$$\partial_r(\sqrt{-g}T_{t(4)}^{\ r}) + \partial_\theta(\sqrt{-g}T_{t(4)}^{\ \theta}) = 0, \tag{4.3.17}$$

where $\sqrt{-g} = (r^2 + a^2\cos^2\theta)\sin\theta$. By integrating Eq. (4.3.17) over the angular coordinates $\theta$ and $\varphi$, we obtain

$$\partial_r T_{t(2)}^{\ r} = 0, \tag{4.3.18}$$

where we define the effective 2-dimensional tensor $T_{t(2)}^{\ r}$ by

$$T_{t(2)}^{\ r} \equiv \int d\Omega_{(2)}(r^2 + a^2\cos^2\theta)T_{t(4)}^{\ r}. \tag{4.3.19}$$

We define $T_{t(2)}^{\ r} \equiv T_{t(O)}^{\ r}$ to emphasize region $O$ far from the horizon. The energy-momentum tensor $T_{t(O)}^{\ r}$ is conserved in region $O$;

$$\partial_r T_{t(O)}^{\ r} = 0. \tag{4.3.20}$$

By integrating Eq. (4.3.20), we obtain

$$T_{t(O)}^{\ r} = a_o, \tag{4.3.21}$$

where $a_o$ is an integration constant.

Finally, we consider the Ward identity for the general coordinate transformation in region $H$ near the horizon. The Ward identity for the general coordinate transformation in the presence of a gravitational anomaly is

$$\nabla_\nu T_{\mu(H)}^{\ \nu}(r) - F_{\mu\nu} J_{(H)}^{\nu}(r) - \frac{\partial_\mu \Phi}{\sqrt{-g_{(2)}}} \frac{\delta S_{(H)}}{\delta \Phi} - \mathscr{A}_\mu(r) = 0, \tag{4.3.22}$$

where both of the gauge current and the energy-momentum tensor are defined to be of the *covariant* form and $\mathscr{A}_\mu$ is the covariant form of the 2-dimensional gravitational anomaly. This Ward identity corresponds to that of [76] when there is no dilaton field. The covariant form of the 2-dimensional gravitational anomaly $\mathscr{A}_\mu$ is given by [82–84]

$$\mathscr{A}_\mu = \frac{1}{96\pi\sqrt{-g_{(2)}}} \epsilon_{\mu\nu} \partial^\nu R = \partial_r N_\mu^{\ r}, \tag{4.3.23}$$

where we define $N_\mu^{\ r}$ by

$$N_t^{\ r} \equiv \frac{f f'' - (f')^2/2}{96\pi}, \qquad N_r^{\ r} \equiv 0, \tag{4.3.24}$$

and $\{\ '\ \}$ represents differentiation with respect to $r$. The $\mu = t$ component of (4.3.22) is written as

$$\partial_r T_{t(H)}^{\ r}(r) = F_{rt} J_{(H)}^r(r) + \partial_r N_t^{\ r}(r). \tag{4.3.25}$$

Using (4.3.14) and integrating (4.3.25) over $r$ from $r_+$ to $r$, we obtain

$$T_{t(H)}^{\ r}(r) = -\frac{m^2}{2\pi} U_t(r_+) U_t(r) + \frac{m^2}{4\pi} U_t^2(r) + N_t^{\ r}(r) + \frac{m^2}{4\pi} U_t^2(r_+) - N_t^{\ r}(r_+), \tag{4.3.26}$$

where we impose the condition that the energy-momentum tensor vanishes at the horizon, which is the same condition as (4.3.15):

$$T_{t(H)}^{\ r}(r_+) = 0. \tag{4.3.27}$$

We compare (4.3.21) with (4.3.26). By following Banerjee and Kulkarni's approach [78], we impose the condition that the asymptotic form of (4.3.26) in the limit $r \to \infty$ is equal to (4.3.21):

$$T^{\ r}_{t(O)} = T^{\ r}_{t(H)}(\infty). \tag{4.3.28}$$

Condition (4.3.28) corresponds to the statement that no energy flux is generated away from the horizon region. Therefore, the asymptotic form of (4.3.26) has to agree with that of (4.3.21). From (4.3.28), we obtain

$$a_o = \frac{m^2\Omega_{\mathrm{H}}^2}{4\pi} + \frac{\pi}{12\beta^2}, \tag{4.3.29}$$

where $\Omega_{\mathrm{H}}$ is the angular velocity of the black hole,

$$\Omega_{\mathrm{H}} \equiv \frac{a}{r_+^2 + a^2}, \tag{4.3.30}$$

and we used both of the surface gravity of the black hole,

$$\kappa = \frac{2\pi}{\beta} = \frac{1}{2}f'(r_+), \tag{4.3.31}$$

and (4.3.24). As a result, we obtain the flux of the energy-momentum tensor in the region far from the horizon from (4.3.28) as

$$T^{\ r}_{t(O)} = \frac{m^2\Omega_{\mathrm{H}}^2}{4\pi} + \frac{\pi}{12\beta^2}. \tag{4.3.32}$$

This flux agrees with the Hawking flux. Our result corresponds to that of [5] in the limit $r \to \infty$. In contrast with the case in [5], our result does not depend on gauge fields in the region far from the horizon where the radial coordinate $r$ is large but finite. As can be seen from the action (4.3.5), the gauge field does not exist in the Kerr black hole physics in a realistic 4-dimensional sense, and only the mass and angular momentum appear. We thus consider that our picture presented here is more natural than that of [5].

### 4.3.2 Comparison with previous works

In this subsection, we would like to state explicitly the differences among our approach and previous works. When one compares our method with that of Iso, Umetsu and Wilczek [5], one recognizes the following differences. To begin with, they define the gauge current by the $\varphi$ component of the 4-dimensional energy-momentum tensor $T^{\ r}_{\varphi(4)}$ in the region far from the horizon. In contrast, we do not define the gauge current in the region far from the horizon, since no gauge

current exists in a Kerr black hole. This difference appears in the energy conservation condition; we use (4.3.20), whereas Iso, Umetsu and Wilczek used the equation

$$\partial_r T_{t(2)}^{\ r} - F_{rt} J_{(2)}^r = 0. \tag{4.3.33}$$

If we define gauge currents suitably, we might be able to consider the Kerr black hole in the same way as the Reissner-Nordström black hole, as Iso, Umetsu and Wilczek attempt to do. However, some subtle aspects are involved in such attempts to define gauge currents.

To be explicit, the authors in [5] regard part of the metrics as the gauge field by defining $A^\mu \equiv -g_{(4)}^{\mu\varphi}$, as in Kaluza-Klein theory. This definition is consistent with the initial definition of the gauge field (4.3.10) near the horizon, i.e.,

$$A_t = \frac{g_{t\varphi(4)}}{g_{\varphi\varphi(4)}} = \frac{a(r^2 + a^2 - \Delta)}{a^2 \Delta \sin^2\theta - (r^2 + a^2)^2} \overset{\Delta \to 0}{\longrightarrow} U_t = -\frac{a}{r^2 + a^2}. \tag{4.3.34}$$

To maintain consistency, they simultaneously assume that the definition of (4.3.19) is modified such that it leads to (4.3.33) by using the $\mu = t$ component of (4.3.16), i.e.,

$$T_{t(2)}^{\ r} = \int d\Omega_2 \left(r^2 + a^2 \cos^2\theta\right) \left(T_{t(4)}^{\ r} - U_t T_{\varphi(4)}^{\ r}\right). \tag{4.3.35}$$

In this way they maintain consistency. However, we consider that definition (4.3.19) is more natural than this modified definition, since in definition (4.3.19) the formal 2-dimensional energy-momentum tensor is defined by integrating the exact 4-dimensional energy-momentum tensor over the angular coordinates without introducing an artificial gauge current in the region far from the horizon. In our approach, which is natural for the Kerr black hole, no gauge field appears in the region far from the horizon where the radial coordinate $r$ is large but finite, in contrast to the formulation in [5]. We thus believe that our formulation is more natural than the formulation in [5], although both formulations give rise to the same physical conclusion.

Furthermore, in comparison between the derivation of Iso, Umetsu and Wilczek and ours, there are important differences. Since they defined the gauge field away from the horizon, they can treat the gauge current there. In the region away from the horizon, the current is conserved

$$\partial_r J_{(O)}^r = 0. \tag{4.3.36}$$

On the contrary, in the near horizon region, the current obeys an anomalous equation

$$\partial_r J_{(O)}^r = \frac{m^2}{4\pi} \partial_r U_t. \tag{4.3.37}$$

The right-hand side is the gauge anomaly in a consistent form [86]. The current is accordingly a consistent current which can be obtained from the variation of the effective action with respect with the gauge field. One can solve these equations in each region as

$$J^r_{(O)} = c_o, \tag{4.3.38}$$

$$J^r_{(H)} = c_H + \frac{m^2}{4\pi}\left(U_t(r) - U_t(r_+)\right), \tag{4.3.39}$$

where $c_o$ and $c_H$ are integration constants.

Here, the authors in [5] consider the effective action $W$ without the ingoing modes in the near horizon. The variation of the effective action under the gauge transformation is then given by

$$-\delta W = \int d^2x\sqrt{-g_{(2)}}\lambda\nabla_\mu J^\mu, \tag{4.3.40}$$

where $\lambda$ is a gauge parameter and we note that all the currents are *consistent* forms. Since the effective theories are different near and far from the horizon, they wrote the current as a sum in two regions

$$J^r = J^r_{(O)}\Theta_+(r) + J^r_{(H)}H(r), \tag{4.3.41}$$

where $\Theta_+(r)$ and $H(r)$ are step functions defined by

$$\Theta_+(r) \equiv \theta(r - r_+ - \epsilon), \tag{4.3.42}$$

$$H(r) \equiv 1 - \Theta_+(r). \tag{4.3.43}$$

By substituting (4.3.41) into (4.3.40) and integrating by parts, we have

$$-\delta W = \int d^2x\lambda\left[\delta(r - r_+ - \epsilon)\left(J^r_{(O)} - J^r_H + \frac{m^2}{4\pi}U_t\right) + \partial_r\left(\frac{m^2}{4\pi}U_tH(r)\right)\right]. \tag{4.3.44}$$

Both the coefficient of the delta function in the first term and the second term should vanish because the total effective action must be gauge invariant. They have required that the second term should be cancelled by quantum effects of the classically irrelevant ingoing modes related to the Wess-Zumino term. By imposing the condition that the coefficient of the *covariant* current at the horizon should vanish, they determined the current flux. Similarly the energy-momentum tensor can also be determined.

We would like to note that they needed the quantum effects of the once ignored ingoing modes because they used step functions for the continuously connected two regions. Also they used the

boundary condition for the covariant current in order to fix the value of the current, and they hence used two kinds of currents (consistent and covariant), which complicate the analysis.

In other approaches, Banerjee and Kulkarni used the Ward identities for the covariant current [76]. However they had to define the consistent current in order to use the Wess-Zumino terms. The Wess-Zumino terms are also used in the approach of Iso, Umetsu and Wilczek [5]. Therefore, their approach is not completely described by covariant currents only. They also considered an approach without step functions [78]. They obtained the Hawking flux by using effective actions and two boundary conditions for the covariant current. However, they assumed that the effective actions are 2-dimensional in both the region near the horizon the region far from the horizon. As discussed in the paper of Iso, Umetsu and Wilczek, the effective theory should be 4-dimensional in the region far from the horizon. If one should assume this 4-dimensional theory as an effective 2-dimensional theory (in the sense of conformal field theory), one encounters a difficulty since one cannot consider matter fields with mass and interactions away from the horizon in conformal field theory according to our current understanding of conformal field theory.

In contrast with the above approaches, we do not use the consistent current at any stage of our analysis since we use neither the Wess-Zumino term nor the effective action. Thus we only use the covariant current. We now argue why we use the regularity conditions for *covariant* currents instead of consistent currents. All the physical quantities should be gauge-invariant. Thus, physical currents should be *covariant*. This is consistent with, for example, the well-known anomalous baryon number current in the Weinberg-Salam theory [85]. Since we do not use any step function either, we need not consider the quantum effect of the ingoing modes. Furthermore, in principle, we can incorporate matter fields with mass and interactions away from the horizon. Therefore, we believe that our approach clarifies some essential aspects of the derivation of Hawking flux from anomalies.

We have shown that the Ward identities and boundary conditions for covariant currents, without referring to the Wess-Zumino terms and the effective action, are sufficient to derive Hawking radiation. The first boundary condition states that both the $U(1)$ gauge current and the energy-momentum tensor vanish at the horizon, as in (4.3.15) and (4.3.27). This condition corresponds to the regularity condition that a free falling observer sees a finite amount of the charged current at the horizon. The second boundary condition is that the asymptotic form of the energy-momentum tensor which was originally defined in the region near the horizon is equal to the energy-momentum tensor in the region far from the horizon in the limit $r \to \infty$, as in (4.3.28). This condition means

that no energy flux is generated away from the near horizon region.

In passing, we mention that the Hawking flux is determined from (4.3.26) simply by considering the direct limit

$$T_{t(H)}^{\ r}(r \to \infty) = \frac{m^2}{4\pi} A_t^2(r_+) - N_t^{\ r}(r_+), \tag{4.3.45}$$

which agrees with (4.3.32). The physical meaning of this consideration is that Hawking radiation is induced by quantum anomalies, which are defined in an arbitrarily small region near the horizon since they are short-distance phenomena, and at any region far from the horizon the theory is anomaly-free and thus, no further flux is generated. Namely, we utilize an intuitive picture on the basis of the Gauss theorem, which is applied to a closed region surrounded by a surface $S$ very close to the horizon and a surface $S'$ far from the horizon in the asymptotic region (Fig. 4.3). If no flux is generated in this closed region, the flux on the surface very close to the horizon and the flux on the surface far from the horizon in the asymptotic region coincide.

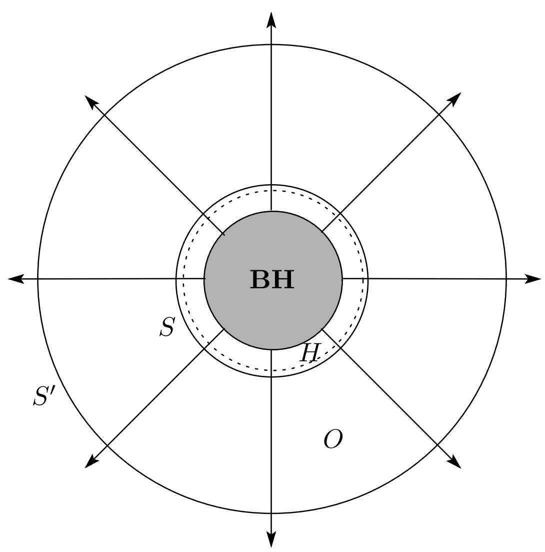


Fig. 4.3 Intuitive picture on the basis of the Gauss theorem. The total fluxes on $S$ and $S'$ are equal by the Gauss theorem.

Finally, we mention that there recently appeared many papers about the derivation of Hawking radiation from anomalies. Further developments associated with this derivation are given in [87–

93]. This method is capable of wide application. For example, it has been extended to various black holes [94–98], and higher spin generalization of the anomaly method have been discussed [99–102].

### 4.3.3 The case of a Reissner-Nordström black hole

In this subsection, we show that Hawking flux in a charged black hole can be obtained by using our approach. Since we consider a charged Reissner-Nordström black hole, the external space is given by the Reissner-Nordström metric

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{4.3.46}$$

and $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}, \tag{4.3.47}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and $r_+$ is the distance from the center of the black hole to the outer horizon. We consider quantum fields in the vicinity of the Reissner-Nordström black hole. In 4 dimensions, the action for a complex scalar field is given by

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} (\partial_\mu + ieV_\mu)\phi^* (\partial_\nu - ieV_\nu)\phi + S_{\rm int}, \tag{4.3.48}$$

where the first term is the kinetic term and the second term $S_{\rm int}$ represents the mass, potential and interaction terms. In contrast with the Kerr black hole background, we note that the $U(1)$ gauge field $V_t = -\dfrac{Q}{r}$ appears in the Reissner-Nordström black hole background. By performing the partial wave decomposition of $\phi$ in terms of the spherical harmonics ($\phi = \sum_{l,m} \phi_{lm} Y_{lm}$) and using the property $f(r_+) = 0$ at the horizon, the action $S_{(H)}$ near the horizon is written as

$$S_{(H)} = -\sum_{l,m} \int dtdr \Phi \phi^*_{lm} \left[ g^{tt}(\partial_t - ieV_t)^2 + \partial_r g^{rr} \partial_r \right] \phi_{lm}, \tag{4.3.49}$$

where we ignore $S_{\rm int}$ because the kinetic term dominates near the horizon in high-energy theory. From this action, we find that $\phi_{lm}$ can be considered as $(1+1)$-dimensional complex scalar fields in the backgrounds of the dilaton $\Phi$, metric $g_{\mu\nu}$ and $U(1)$ gauge field $V_\mu$, where

$$\Phi = r^2, \tag{4.3.50}$$

$$g_{tt} = f(r), \quad g_{rr} = -\frac{1}{f(r)}, \quad g_{rt} = 0, \tag{4.3.51}$$

$$V_t = -\frac{Q}{r}, \quad V_r = 0. \tag{4.3.52}$$

The $U(1)$ charge of the 2-dimensional field $\phi_{lm}$ is $e$. Note that the action in the region far from the horizon is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, V_\mu]$ and the action in the region near the horizon is $S_{(H)}[\phi, g^{\mu\nu}_{(2)}, V_\mu, \Phi]$.

We now present the derivation of Hawking radiation for the Reissner-Nordström black hole. First, we consider the Ward identity for the gauge transformation in region $O$ far away from the horizon. Here, we formally perform the path integral for $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, V_\mu]$, where the Nöther current is constructed by the variational principle. Therefore, we can naturally treat *covariant* currents [73]. As a result, we obtain the Ward identity

$$\nabla_\mu J^\mu_{(4)} = 0, \tag{4.3.53}$$

where $J^\mu_{(4)}$ is the 4-dimensional gauge current. Since the Reissner-Nordström background is stationary and spherically symmetric, the expectation value of the gauge current in the background depends only on $r$, i.e., $\langle J^\mu \rangle = \langle J^\mu(r) \rangle$. Using the 4-dimensional metric, the conservation law (4.3.53) is written as

$$\partial_r(\sqrt{-g} J^r_{(4)}) + (\partial_\theta \sqrt{-g}) J^\theta_{(4)} = 0, \tag{4.3.54}$$

where $\sqrt{-g} = r^2 \sin\theta$. By integrating Eq. (4.3.54) over the angular coordinates $\theta$ and $\varphi$, we obtain

$$\partial_r J^r_{(2)} = 0, \tag{4.3.55}$$

where we define the effective 2-dimensional current $J^r_{(2)}$ by

$$J^r_{(2)} \equiv \int d\Omega_{(2)} r^2 J^r_{(4)}. \tag{4.3.56}$$

We define $J^r_{(2)} \equiv J^r_{(O)}$ to emphasize region $O$ far from the horizon. The gauge current $J^r_{(O)}$ is conserved in region $O$,

$$\partial_r J^r_{(O)} = 0. \tag{4.3.57}$$

By integrating Eq. (4.3.57), we obtain

$$J^r_{(O)} = c_o, \tag{4.3.58}$$

where $c_o$ is an integration constant.

Second, we consider the Ward identity for the gauge transformation in the region $H$ near the horizon. When there is a gauge anomaly, the Ward identity for the gauge transformation is given by

$$\nabla_\mu J^\mu_{(H)} - \mathscr{B} = 0, \tag{4.3.59}$$

where we define the covariant current as $J^{\mu}_{(H)}$ and $\mathscr{B}$ is a covariant gauge anomaly. The covariant form of the 2-dimensional gauge anomaly $\mathscr{B}$ is given by

$$\mathscr{B} = \pm \frac{e^2}{4\pi\sqrt{-g_{(2)}}} \epsilon^{\mu\nu} F_{\mu\nu}, \qquad (\mu, \nu = t, r) \tag{4.3.60}$$

where $+(-)$ corresponds to the anomaly for right(left)-handed fields. Here $\epsilon^{\mu\nu}$ is an antisymmetric tensor with $\epsilon^{tr} = 1$ and $F_{\mu\nu}$ is the field-strength tensor defined by

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu. \tag{4.3.61}$$

Using the 2-dimensional metric (4.3.51), (4.3.59) is written as

$$\partial_r J^r_{(H)}(r) = \frac{e^2}{2\pi} \partial_r V_t(r). \tag{4.3.62}$$

By integrating (A.16) over $r$ from $r_+$ to $r$, we obtain

$$J^r_{(H)}(r) = \frac{e^2}{2\pi} [V_t(r) - V_t(r_+)], \tag{4.3.63}$$

where we impose the condition

$$J^r_{(H)}(r_+) = 0. \tag{4.3.64}$$

This condition corresponds to (4.3.15) in the present paper. We also impose the condition that the asymptotic form of (4.3.63) is equal to that of (4.3.58),

$$J^r_{(O)}(\infty) = J^r_{(H)}(\infty). \tag{4.3.65}$$

From (4.3.65), we obtain the gauge current in region $O$ as

$$J^r_{(O)} = -\frac{e^2}{2\pi} V_t(r_+). \tag{4.3.66}$$

Third, we consider the Ward identity for the general coordinate transformation in the region $O$ far from the horizon. We define the formal 2-dimensional energy-momentum tensor in region $O$ from the exact 4-dimensional energy-momentum tensor in region $O$ and we connect the 2-dimensional energy-momentum tensor in region $O$ with the 2-dimensional energy-momentum tensor thus defined in region $H$. Since the action is $S_{(O)}[\phi, g^{\mu\nu}_{(4)}, V_\mu]$ in region $O$, the Ward identity for the general coordinate transformation, which is anomaly-free is written as

$$\nabla_\nu T_{\mu(4)}^{\ \nu} - F_{\nu\mu} J^{\nu}_{(4)} = 0, \tag{4.3.67}$$

where $T^{\mu\nu}_{(4)}$ is the 4-dimensional energy-momentum tensor. Since the Reissner-Nordström background is stationary and spherically symmetric, the expectation value of the energy-momentum tensor in the background depends only on $r$, i.e., $\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}(r)\rangle$. The $\mu = t$ component of the conservation law (4.3.67) is written as

$$\partial_r \left(\sqrt{-g} T_{t(4)}^{\ r}\right) + \left(\partial_\theta \sqrt{-g}\right) T_{t(4)}^{\ \theta} - \sqrt{-g} F_{rt} J^r_{(4)} = 0. \tag{4.3.68}$$

By integrating (4.3.68) over $\theta$ and $\varphi$, we obtain

$$\partial_r T_{t(2)}^{\ r} = F_{rt} J^r_{(2)}, \tag{4.3.69}$$

where we define the effective 2-dimensional tensor $T_{t(2)}^{\ r}$ by

$$T_{t(2)}^{\ r} \equiv \int d\Omega_{(2)} r^2 T_{t(4)}^{\ r}, \tag{4.3.70}$$

and $J^r_{(2)}$ is defined by (4.3.56). To emphasize region $O$ far from the horizon, we write (4.3.69) as

$$\partial_r T_{t(O)}^{\ r} = F_{rt} J^r_{(O)}. \tag{4.3.71}$$

By substituting (4.3.66) into (4.3.71) and integrating it over $r$, we obtain

$$T_{t(O)}^{\ r}(r) = a_o - \frac{e^2}{2\pi} V_t(r_+) V_t(r). \tag{4.3.72}$$

Finally, we consider the Ward identity for the general coordinate transformation in region $H$ near the horizon. When there exists a gravitational anomaly, the Ward identity for the general coordinate transformation is given by

$$\nabla_\nu T_{\mu(H)}^{\ \nu} - F_{\nu\mu} J^\nu_{(H)} - \frac{\partial_\mu \Phi}{\sqrt{-g}} \frac{\delta S}{\delta \Phi} - \mathscr{A}_\mu = 0, \tag{4.3.73}$$

where both the gauge current and the energy-momentum tensor are defined to be of the *covariant* form and $\mathscr{A}_\mu$ is the covariant form of the 2-dimensional gravitational anomaly. This Ward identity corresponds to that of [76] when there is no dilaton field. The covariant form of the 2-dimensional gravitational anomaly $\mathscr{A}_\mu$ agrees with (4.3.23). Using the 2-dimensional metric (4.3.51), the $\mu = t$ component of (4.3.73) is written as

$$\partial_r T_{t(H)}^{\ r}(r) = \partial_r \left[ -\frac{e^2}{2\pi} V_t(r_+) V_t(r) + \frac{e^2}{4\pi} V_t^2(r) + N_t^{\ r} \right]. \tag{4.3.74}$$

By integrating (4.3.74) over $r$ from $r_+$ to $r$, we obtain

$$T_{t(H)}^{\ r}(r) = -\frac{e^2}{2\pi} V_t(r_+) V_t(r) + \frac{e^2}{4\pi} V_t^2(r) + N_t^{\ r}(r) + \frac{e^2}{4\pi} V_t^2(r_+) - N_t^{\ r}(r_+), \tag{4.3.75}$$

where we impose the condition that the energy-momentum tensor vanishes at the horizon, which is the same as (4.3.27):

$$T_{t(H)}^{\ r}(r_+) = 0. \tag{4.3.76}$$

As for (4.3.28), we impose the condition that the asymptotic form of (4.3.75) in the limit $r \to \infty$ is equal to that of (4.3.72),

$$T_{t(O)}^{\ r}(\infty) = T_{t(H)}^{\ r}(\infty). \tag{4.3.77}$$

From (4.3.77), we obtain

$$a_o = \frac{e^2}{4\pi} V_t^2(r_+) - N_t^{\ r}(r_+). \tag{4.3.78}$$

We thus obtain the flux of the energy-momentum tensor in the region far from the horizon as

$$T_{t(O)}^{\ r}(r) = \frac{e^2 Q^2}{4\pi r_+^2} + \frac{\pi}{12\beta^2} + \frac{e^2 Q}{2\pi r_+} V_t(r). \tag{4.3.79}$$

This result agrees with that of [4]. In contrast with the case of a rotating Kerr black hole, the energy flux depends on the gauge field in the region far from the horizon, where the radial coordinate $r$ is large but still finite, since the gauge field exists in a charged Reissner-Nordström black hole background. However, in the evaluation of Hawking radiation by setting $r \to \infty$, the effect of the gauge field $V_t(r)$ disappears.

# Chapter 5

# Hawking Radiation and Tunneling Mechanism

Parikh and Wilczek proposed a method of deriving Hawking radiation based on quantum tunneling [7]. This derivation using the tunneling mechanism is intuitive and it is also capable of wide application. The essential idea of the tunneling mechanism is that a particle-antiparticle pair is formed close to the horizon. The ingoing mode is trapped inside the horizon while the outgoing mode can quantum mechanically tunnel through the horizon and it is observed at infinity as the Hawking flux (Fig. 5.1). As a background of this derivation, we might consider that for the outgoing particles inside a black hole, the horizon plays a role as an infinite barrier. This infinite barrier may be written as a potential of the delta-function type. The particles cannot classically pass through the potential. According to quantum theory, it is well known that a part of particles can pass through the potential by the quantum tunneling effect. By applying the above discussion to the case of a black hole, we can regard the particles appeared outside the horizon as the radiation from the black hole. Since both of the tunneling effect and Hawking radiation are typical quantum effects, it is note that the quantum tunneling effect is related to Hawking radiation.

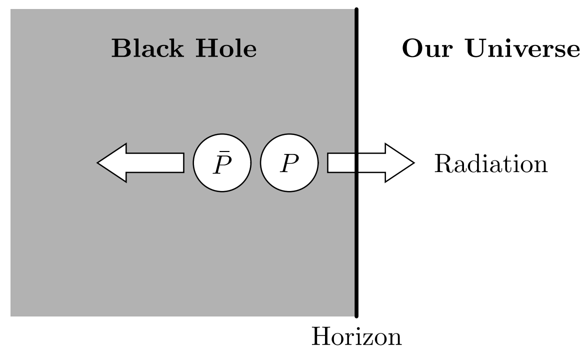


Fig. 5.1 Intuitive picture of the tunneling mechanism

Parikh and Wilczek calculated the WKB amplitudes for classically forbidden paths. The first order calculation is given by

$$\Gamma \sim e^{-2\mathrm{Im}S} \sim e^{-\frac{2\pi\omega}{\kappa}}, \tag{5.0.1}$$

where $\Gamma$ is the tunneling probability, $S$ is the action of the system, $\omega$ is a frequency and $\kappa$ is the surface gravity of the black hole respectively. In comparison with the Boltzmann factor in a thermal equilibrium state at a temperature $\mathcal{T}$,

$$\Gamma_{\mathrm{B}} = e^{-\frac{\omega}{\mathcal{T}}}, \tag{5.0.2}$$

it is confined that the temperature of (5.0.1) agrees with the Hawking temperature,

$$\mathcal{T}_{\mathrm{BH}} = \frac{\kappa}{2\pi}. \tag{5.0.3}$$

However, the analysis is confined to the derivation of the Hawking temperature only by comparing the tunneling probability of an outgoing particle with the Boltzmann factor. There exists no discussion of the spectrum. Therefore, there remains the possibility that the black hole is not the black but merely the thermal body. This problem was pointed out by Banerjee and Majhi [8]. They directly showed how to reproduce the black body spectrum with the Hawking temperature from the expectation value of number operator by using the properties of the tunneling mechanism. Thus the derivation from the tunneling mechanism became more satisfactory.

Their result is valid for black holes with spherically symmetric geometry such as Schwarzschild or Reissner-Nordström black holes in the 4-dimensional theory. However, 4-dimensional black holes have not only a mass and a charge but also angular momentum according to the black

hole uniqueness theorem (the no hair theorem) [14, 16]. In 4 dimensions, the Kerr-Newman black hole, which has both the charge and angular momentum, is the most general black hole and its geometry becomes spherically asymmetric because of its own rotation. There exist several previous works for a rotating black hole in the tunneling method (see for example [103–107]), but they are mathematically very involved.

We would like to extend the simplified derivation of Hawking radiation by Banerjee and Majhi on the basis of the tunneling mechanism to the case of the Kerr-Newman black hole. In Section 2.4, we have shown that the 4-dimensional Kerr-Newman metric effectively becomes a 2-dimensional spherically symmetric metric by using the technique of the dimensional reduction near the horizon. This technique was often used in the derivation of Hawking radiation from anomalies [5, 6].

We note that this technique is valid only for the region very close to the horizon. The use of the same technique in the tunneling mechanism is justified since the tunneling effect is also the quantum effect arising within the Planck length near the horizon region. By this procedure, the metric for the Kerr-Newman black hole becomes an effectively 2-dimensional spherically symmetric metric, and we can use the approach of Banerjee and Majhi which is valid for black holes with spherically symmetric geometry. We can thus derive the black body spectrum and Hawking flux for the Kerr-Newman black hole in the tunneling mechanism.

The contents of this chapter are as follows. In Section 5.1, we review the derivation of black hole radiation by the tunneling mechanism due to Parikh and Wilczek. In Section 5.2, we discuss the method of Parikh and Wilczek from a point of view of the canonical theory. In Section 5.3, we review a variant of the derivation from the tunneling mechanism by Banerjee and Majhi. In Section 5.4, we extend the method of Banerjee and Majhi to the case of a Kerr-Newman black hole.

## 5.1 Hawking Radiation as Tunneling

Parikh and Wilczek proposed a method of deriving Hawking radiation based on quantum tunneling [7]. In this section, we would like to review the derivation by Parikh and Wilczek. For sake of simplicity, we consider the case of a 4-dimensional Schwarzschild black hole. It is well known that the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\Omega^2, \tag{5.1.1}$$

where $d\Omega^2$ is a 2-dimensional unit sphere. This metric is singular at $r = 0$ and $r = 2M$. A singularity at $r = 0$ is the curvature singularity which cannot be removed, while the other singularity at $r = 2M$ is a fictitious singularity arising merely from an improper choice of coordinates.

To remove the fictitious singularity, Parikh and Wilczek introduced the Painlevé coordinates [108]

$$t_p = t + 2\sqrt{2Mr} + 2M \ln \left( \frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}} \right), \tag{5.1.2}$$

with

$$dt_p = dt + \sqrt{\frac{2M}{r}} \frac{1}{1 - \frac{2M}{r}} dr. \tag{5.1.3}$$

By substituting (5.1.3) into (5.1.1), the Painlevé metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_p^2 + 2\sqrt{\frac{2M}{r}} dt_p dr + dr^2 + r^2 d\Omega^2, \tag{5.1.4}$$

and we can confirm that there is no singularity at $r = 2M$. The radial null geodesics are given by

$$\dot{r}_p \equiv \frac{dr}{dt_p} = \pm 1 - \sqrt{\frac{2M}{r}}, \tag{5.1.5}$$

where the positive (negative) sign corresponds to the outgoing (ingoing) geodesic, under the implicit assumption that $t_p$ increases towards the future.

These equations are modified when the particle's self-gravitation is taken into account. Self-gravitating shells in Hamiltonian gravity were studied by Kraus and Wilczek [109]. Now we consider that a particle with a positive energy $\omega$ inside a black hole quantum mechanically tunnels through the horizon and it appears outside the black hole (Fig. 5.2). By the energy conservation law, the black hole energy decreases when the particle escapes from the black hole, namely,

$$M = (M - \omega) + \omega, \tag{5.1.6}$$

where $M$ is the total ADM mass [110–112] of the initial black hole, the first term $(M - \omega)$ of the right-hand side is the mass of the final black hole and the second term $\omega$ is the energy of particle. After the particle escapes from the black hole, the metric of the black hole is given by

$$ds^2 = -\left(1 - \frac{2(M-\omega)}{r}\right) dt_p^2 + 2\sqrt{\frac{2(M-\omega)}{r}} dt_p dr + dr^2 + r^2 d\Omega^2, \tag{5.1.7}$$

from (5.1.4). Strictly speaking, it seems that we need to consider the time dependence of the black hole mass in this process. However, since it is involute, Parikh and Wilczek used this static metric as the background metric.

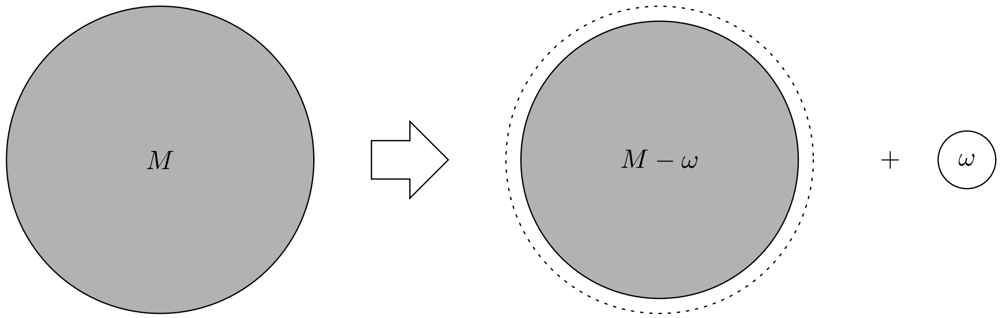


Fig. 5.2 Particle emission by tunneling

Since the typical wavelength of the radiation is of the order of the size of the black hole, one might doubt whether a point particle description is appropriate. However, when the outgoing wave is traced back towards the horizon, its wavelength, as measured by local fiducial observers, is ever-increasingly blue-shifted. Thus they considered that the radial wave number approaches infinity and the point particle or WKB approximation is justified near the horizon.

The imaginary part of the action for an s-wave outgoing positive energy particle which crosses the horizon outwards from $r_{\rm in}$ to $r_{\rm out}$ can be expressed as

$$\mathrm{Im}\ S = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} p_p dr = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_p} dp'_p dr, \tag{5.1.8}$$

where $p_p$ is the canonical momentum for the radial coordinate $r$. By using the Hamilton equation

$$\dot{r}_p = \frac{dH_p}{dp_p}, \tag{5.1.9}$$

where $H_p$ is the Hamiltonian, the relation (5.1.8) becomes

$$\mathrm{Im}\ S = \mathrm{Im}\int_M^{M-\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{1}{\dot{r}_p} dH_p dr. \tag{5.1.10}$$

By substituting the outgoing mode of the radial null geodesics associated with (5.1.7) into (5.1.10) and using the Hamiltonian $H_p = M' - \omega$, we obtain

$$\mathrm{Im}\ S = \mathrm{Im}\int_M^{M-\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{1}{1-\sqrt{\frac{2(M-\omega)}{r}}} dM' dr, \tag{5.1.11}$$

where we regarded $\omega$ as a constant and $M'$ as a variable. By using Feynman's $i\epsilon$ prescription for

positive energy solutions $\omega \to \omega - i\epsilon$, we obtain $M \to M - i\epsilon$ and

$$\text{Im } S = \text{Im} \int_M^{M-\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{1}{1 - \sqrt{\frac{2M'}{r}} + i\epsilon} dM' dr \tag{5.1.12}$$

$$= \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \left[ P \frac{1}{1 - \sqrt{\frac{2M'}{r}}} + \int_M^{M-\omega} -i\pi\delta \left( 1 - \sqrt{\frac{2M'}{r}} \right) dM' \right] dr, \tag{5.1.13}$$

where $P$ stands for the principal value. We can ignore the first term because it is a real part. Now since the mass ranges from $M$ to $M - \omega$, the radial coordinate ranges from $r_{\text{in}} = 2M$ to $r_{\text{out}} = 2(M - \omega)$. Thus we can obtain the imaginary part of the action

$$\text{Im } S = \int_{r_{\text{in}}}^{r_{\text{out}}} \int_M^{M-\omega} -\pi\delta \left( 1 - \sqrt{\frac{2M'}{r}} \right) dM' dr \tag{5.1.14}$$

$$= \int_{2M}^{2(M-\omega)} (-\pi r) dr \tag{5.1.15}$$

$$= 4\pi\omega M - 2\pi\omega^2. \tag{5.1.16}$$

The radially inward motion has a classically forbidden trajectory because the apparent horizon is itself contracting. Thus, the limits on the integral indicate that, over the course of the classically forbidden trajectory, the outgoing particle starts from $r = 2M - \epsilon$, just inside the initial position of the horizon, and traverses the contracting horizon to materialize at $r = 2(M - \omega) + \epsilon$, just outside the final position of the horizon (Fig. 5.3).

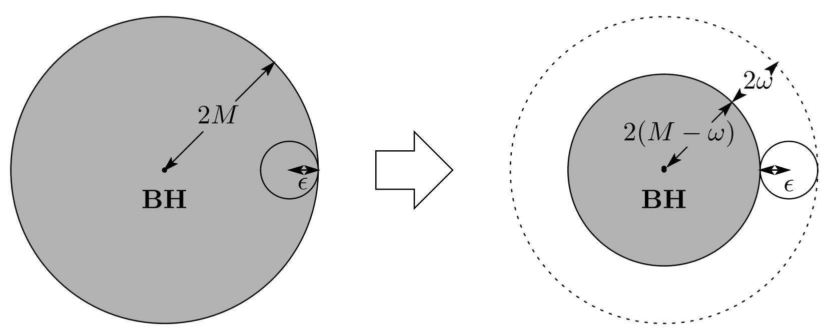


Fig. 5.3 The contracting horizon

By using (5.1.16), we find the semi-classical WKB probability as

$$\Gamma \sim e^{-2\text{Im } S} = e^{-8\pi\omega M + 4\pi\omega^2}. \tag{5.1.17}$$

Note that the probability is given by an absolute square of the amplitude. When we can ignore the quadratic term of $\omega$ in the case of $M \gg \omega$, the probability (5.1.17) can be written as

$$\Gamma \sim e^{-8\pi\omega M} = e^{-\frac{2\pi\omega}{\kappa}}, \tag{5.1.18}$$

where we used the surface gravity of the Schwarzschild black hole $\kappa = \dfrac{1}{4M}$. Here we recall thermodynamics. It is well known that the Boltzmann factor in a thermal equilibrium state at a temperature $\mathcal{T}$ is given by

$$\Gamma_{\mathrm{B}} = e^{-\frac{\omega}{\mathcal{T}}}. \tag{5.1.19}$$

By comparing between (5.1.18) and (5.1.19), we find that the temperature of the black hole is obtained by

$$\mathcal{T}_{\mathrm{BH}} = \frac{\kappa}{2\pi}. \tag{5.1.20}$$

This result agrees with the result of previous works as in (3.1.75).

## 5.2 Tunneling Mechanism in the Canonical Theory

The tunneling mechanism by Parikh and Wilczek is explicit and straightforward in the canonical theory. In this subsection, we would like to review the method of Parikh and Wilczek by using the canonical theory.

To begin with, we consider the action of the system. The action is defined by

$$S = \int L dt = -\mu_m \int ds = \int -\mu_m \frac{ds}{dt} dt, \tag{5.2.1}$$

where $L$ is Lagrangian defined by

$$L = -\mu_m \frac{ds}{dt}, \tag{5.2.2}$$

and $\mu_m$ is a mass of a particle. It is convenient to take the time component of the Painlevé metric positive, namely,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt_p^2 - 2\sqrt{\frac{2M}{r}} dt_p dr - dr^2, \tag{5.2.3}$$

where the Painlevé time $t_p$ is defined by

$$t_p = t + 2\sqrt{2Mr} + 2M \ln\left(\frac{\sqrt{r} - \sqrt{2M}}{\sqrt{r} + \sqrt{2M}}\right), \tag{5.2.4}$$

and $t$ is the Schwarzschild time. From the metric (5.2.3), we obtain

$$\frac{ds}{dt_p} = \pm\sqrt{\left(1-\frac{2M}{r}\right) - 2\sqrt{\frac{2M}{r}}\frac{dr}{dt_p} - \left(\frac{dr}{dt_p}\right)^2}, \tag{5.2.5}$$

where we adopt $+$. We thus obtain the Lagrangian

$$L_p(r,\dot{r}_p) = -\mu_m\sqrt{\left(1-\frac{2M}{r}\right) - 2\sqrt{\frac{2M}{r}}\dot{r}_p - \dot{r}_p^2}, \tag{5.2.6}$$

where $\dot{r}_p \equiv \dfrac{dr}{dt_p}$. The canonical momentum for the radial coordinate $r$ is defined by

$$p_p \equiv \frac{\partial L_p}{\partial \dot{r}_p} = \frac{\mu_m\left(\dot{r}_p + \sqrt{\frac{2M}{r}}\right)}{\sqrt{1-\left(\dot{r}_p + \sqrt{\frac{2M}{r}}\right)^2}}, \tag{5.2.7}$$

and by solving for $\dot{r}_p$, we obtain

$$\dot{r}_p = \pm\sqrt{\frac{p_p^2}{p_p^2+\mu_m^2}} - \sqrt{\frac{2M}{r}}, \tag{5.2.8}$$

where $+(-)$ represents the outgoing (ingoing) mode. We adopt $+$, because we consider the outgoing mode.

In canonical theory, the Hamiltonian is defined by

$$H = p\dot{r} - L(r,\dot{r}). \tag{5.2.9}$$

By substituting (5.2.8) into (5.2.9), we obtain

$$H_p(r,p_p) = p_p\left(\sqrt{\frac{p_p^2}{p_p^2+\mu_m^2}} - \sqrt{\frac{2M}{r}}\right) + \mu_m\sqrt{\frac{\mu_m^2}{p_p^2+\mu_m^2}}. \tag{5.2.10}$$

In the derivation of Parikh and Wilczek, they used the null geodesic equation. This can be reproduced by taking $\mu_m = 0$ in (5.2.8). By substituting $\mu_m = 0$ into the Hamiltonian (5.2.10) and solving for $p_p$, we obtain

$$p_p = \frac{H_p}{1-\sqrt{\frac{2M}{r}}}. \tag{5.2.11}$$

From both (5.2.9) and (5.2.11), the action (5.2.1) is written as

$$S = \int L_p dt_p = \int \left[p_p\dot{r}_p - H_p\right]dt_p = \int p_p dr - \int H_p dt_p = \int \frac{H_p}{1-\sqrt{\frac{2M}{r}}}dr - \int H_p dt_p. \tag{5.2.12}$$

Now since the metric is stationary, it has a time-like Killing vector and there exists an energy as the corresponding conserved quantity. The energy is defined as $\omega$ and it is the eigenvalue of the Hamiltonian $H_p = \omega$. The action (5.2.12) is written as

$$S = \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\omega}{1 - \sqrt{\frac{2M}{r}}} dr - \int_{t_{p(\text{in})}}^{t_{p(\text{out})}} \omega dt_p. \tag{5.2.13}$$

Now, we consider that an outgoing positive energy particle arising by the pair creation at $r_{\text{in}}$ close to the horizon inside the black hole, appears at $r_{\text{out}}$ close to the horizon outside the black hole through the horizon $r_{\text{H}} = 2M$. The Painlevé-time coordinates corresponding to these coordinates are respectively defined as $t_{p(\text{in})}$ and $t_{p(\text{out})}$.

Since Hawking radiation is a quantum effect, we have only to evaluate the classically hidden action i.e., the imaginary part of the action

$$\text{Im } S = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\omega}{1 - \sqrt{\frac{2M}{r}}} dr. \tag{5.2.14}$$

In the second term of (5.2.13), the Painlevé-time coordinate is finite on the horizon and it has no discontinuous point between $t_{p(\text{in})}$ and $t_{p(\text{out})}$. This can be understood from a naive discussion that the Painlevé coordinate can be kept to finite values even by substituting the future event horizon $(t, r) = (+\infty, 2M)$ into (5.2.4). We can ignore the second term in (5.2.13) because it gives the real part only. Since the first term in (5.2.13) is singular at the horizon $r = 2M$, there is a possibility that the imaginary part appears, and we need to evaluate it.

By using the Feynman's $i\epsilon$ prescription for a real particle, we can obtain

$$\begin{aligned}
\text{Im } S &= \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\omega}{1 - \sqrt{\frac{2M}{r}} - i\epsilon} dr && (5.2.15) \\
&= \text{Im} \left[ P \frac{\omega}{1 - \sqrt{\frac{2M}{r}}} + i\pi\omega \int_{r_{\text{in}}}^{r_{\text{out}}} \delta\left(1 - \sqrt{\frac{2M}{r}}\right) dr \right] && (5.2.16) \\
&= 4\pi\omega M, && (5.2.17)
\end{aligned}$$

where $P$ stands for the principal value (the real part). This result agrees with (5.1.16) to the first order.

## 5.3 Hawking Black Body Spectrum from Tunneling Mechanism

A method of deriving Hawking radiation based on quantum tunneling was originally proposed by Parikh and Wilczek [7]. After Parikh and Wilczek's derivation, a lot of papers on the tunneling mechanism have been published. However the analysis has been confined to obtain the Hawking temperature only by comparing the tunneling probability of an outgoing particle with the Boltzmann factor. The discussion of the spectrum was not transparent. Therefore, there remains the possibility that the black hole is not the black but merely the thermal body. In this sense the tunneling method, presented so far, is not satisfactory yet. This problem was emphasized by Banerjee and Majhi [8]. They showed how to reproduce the black body spectrum with the Hawking temperature directly from the expectation value of number operator by using the properties of the tunneling mechanism. Thus the derivation by the tunneling mechanism became more satisfactory.

In this subsection, we would like to review Banerjee and Majhi's method [8]. For sake of simplicity, we consider the case of a Schwarzschild black hole background. The metric is then given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{5.3.1}$$

where $f(r)$ is defined by

$$f(r) \equiv 1 - \frac{2M}{r}. \tag{5.3.2}$$

We would like to note that they used not the Painlevé metric (5.1.4) but the Schwarzschild metric (5.3.1).

Here we consider the Klein-Gordon equation for a massless scalar field

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = 0, \tag{5.3.3}$$

where $\nabla_\mu$ is the covariant derivative defined by (3.1.8) and (3.1.9). By using the $(r-t)$ sector of the metric (5.3.1) in (5.3.3),

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{5.3.4}$$

we obtain

$$\left[\frac{1}{f(r)}\partial_t^2 - f(r)\partial_r^2 - f'(r)\partial_r\right]\phi = 0. \tag{5.3.5}$$

Taking the standard WKB ansatz

$$\phi(r,t) = e^{\frac{i}{\hbar}S(r,t)}, \tag{5.3.6}$$

and substituting the expansion for $S(r,t)$ in terms of the Planck constant $\hbar$

$$S(r,t) = S_0(r,t) + \sum_{i=1}^{\infty} \hbar^i S_i(r,t), \tag{5.3.7}$$

in (5.3.5) where we write the Planck constant explicitly, we obtain, in the semiclassical limit (i.e., keeping only $S_0$),

$$\partial_t S_0(r,t) = \pm f(r)\partial_r S_0(r,t). \tag{5.3.8}$$

Now we consider the classical Hamilton-Jacobi equation

$$\frac{\partial S_0}{\partial t} + H = 0, \tag{5.3.9}$$

where $S_0$ is the classical action and $H$ is the Hamiltonian. Since the metric (5.3.4) is stationary, it has a timelike Killing vector. Thus the Hamiltonian is given by

$$H = \omega, \tag{5.3.10}$$

where $\omega$ is a constant and the conserved quantity corresponding to the timelike Killing vector. This is identified as the effective energy experienced by the particle at asymptotic infinity. By the Hamilton-Jacobi equation (5.3.9), we obtain

$$S_0 = -\omega t + \tilde{S}_0(r), \tag{5.3.11}$$

where $\tilde{S}_0(r)$ is a time-independent arbitrary function. By substituting (5.3.11) into (5.3.8), we obtain

$$\omega = \pm f(r)\partial_r \tilde{S}_0(r). \tag{5.3.12}$$

By using the relation of the tortoise coordinate defined by (2.2.6),

$$\partial_{r_*} = f(r)\partial_r, \tag{5.3.13}$$

the Hamilton-Jacobi equation (5.3.12) becomes

$$\partial_{r_*}\tilde{S}_0(r) = \pm\omega. \tag{5.3.14}$$

By integrating (5.3.14) over $r_*$, we obtained

$$\tilde{S}_0(r) = \pm\omega r_* + C, \tag{5.3.15}$$

where $C$ is an integration constant and we ignore it since it is included in a normalization constant of the wave function. By substituting this into (5.3.11), the classical action becomes

$$S_0(r,t) = -\omega(t \mp r_*). \tag{5.3.16}$$

Thus we can obtain the semiclassical solution for the scalar field

$$\phi(r,t) = \exp\left[-\frac{i}{\hbar}\omega(t \mp r_*)\right]. \tag{5.3.17}$$

Here we introduce both the retarded time $u$ and the advanced time $v$ defined by (2.2.9),

$$u \equiv t - r_*, \qquad v \equiv t + r_*, \tag{5.3.18}$$

where we can regard $u$ ($v$) as the outgoing (ingoing) modes of particles. We can then separate the scalar field (5.3.17) into the ingoing (left handed) modes and outgoing (right handed) modes. Since the tunneling effect is the quantum effect arising within the Planck length in the near horizon region, we have to consider both the inside and outside regions which are very close to the horizon. In the regions $r_+ - \varepsilon < r < r_+$, and $r_+ \le r < r_+ - \varepsilon$, respectively, we express the field $\phi$ as

$$\left.\begin{aligned} \phi^R_{\rm in} &= \exp\left(-\frac{i}{\hbar}\omega u_{\rm in}\right) \\ \phi^L_{\rm in} &= \exp\left(-\frac{i}{\hbar}\omega v_{\rm in}\right) \end{aligned}\right\} \qquad (r_+ - \varepsilon < r < r_+), \tag{5.3.19}$$

$$\left.\begin{aligned} \phi^R_{\rm out} &= \exp\left(-\frac{i}{\hbar}\omega' u_{\rm out}\right) \\ \phi^L_{\rm out} &= \exp\left(-\frac{i}{\hbar}\omega' v_{\rm out}\right) \end{aligned}\right\} \qquad (r_+ < r < r_+ + \varepsilon), \tag{5.3.20}$$

where $\varepsilon$ is an arbitrarily small constant, "R (L)" stands for the right (left) modes and "in (out)" stands for the inside (outside) of the black hole, respectively (Fig. 5.4). Here we note that these fields are defined both in the inside and outside regions which are very close to the horizon.

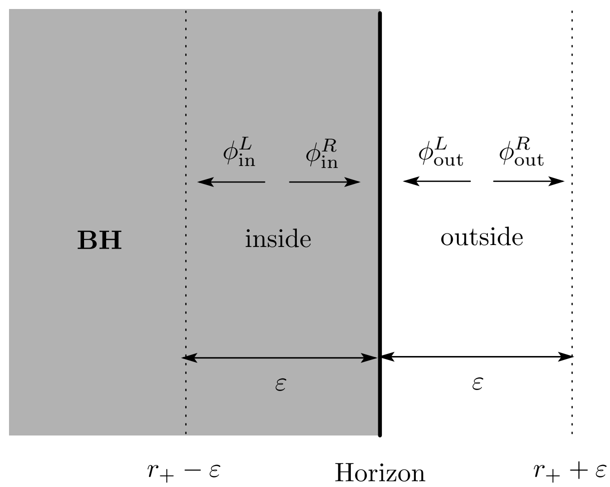


Fig. 5.4 Intuitive picture of the scalar field near the horizon

Now we consider that the outgoing particles inside the black hole quantum mechanically tunnel through the horizon. However, it is well known that both the Schwarzschild coordinates $(t, r)$ and the Eddington-Finkelstein coordinates $(u, v)$ are singular at the horizon. To describe horizon-crossing phenomena, Parikh and Wilczek used the Painlevé coordinates which have no singularity at the horizon as already stated in Subsection 5.1. On the other hand, Banerjee and Majhi used the Kruskal-Szekeres coordinates defined by

$$\left.\begin{aligned} T &= \left(\frac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \\ R &= \left(\frac{r}{2M}-1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \end{aligned}\right\} \quad \text{when} \quad r > 2M, \tag{5.3.21}$$

$$\left.\begin{aligned} T &= \left(1-\frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \\ R &= \left(1-\frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \end{aligned}\right\} \quad \text{when} \quad r < 2M. \tag{5.3.22}$$

It is well known that these coordinates also have no singularity at the horizon. The Kruskal time $T$ and radial $R$ coordinates inside and outside the horizon are represented as

$$\left.\begin{aligned} T_{\rm out} &= \exp\left[\kappa(r_*)_{\rm out}\right] \sinh\left(\kappa t_{\rm out}\right) \\ R_{\rm out} &= \exp\left[\kappa(r_*)_{\rm out}\right] \cosh\left(\kappa t_{\rm out}\right) \end{aligned}\right\}, \tag{5.3.23}$$

$$\left.\begin{aligned} T_{\rm in} &= \exp\left[\kappa(r_*)_{\rm in}\right] \cosh\left(\kappa t_{\rm in}\right) \\ R_{\rm in} &= \exp\left[\kappa(r_*)_{\rm in}\right] \sinh\left(\kappa t_{\rm in}\right) \end{aligned}\right\}, \tag{5.3.24}$$

where we used the tortoise coordinates as in (2.2.8) and the surface gravity of the Schwarzschild black hole (3.1.52) in both (5.3.21) and (5.3.22).

In general, the Schwarzschild metric describes the behavior outside the black hole. Consequently, the readers may wonder if these metrics defined by the Kruskal-Szekeres coordinates can really describe the behavior inside the black hole. In our case, however, we study the tunneling effect across the horizon and thus we study the behavior in the very small regions near the horizon. In such an analysis, due to the reasons of continuity, it may not be unnatural to assume that the Kruskal-Szekeres coordinates (5.3.23) and (5.3.24) can be used in both of outside and inside regions of near the horizon. We note that these coordinates was used in the description of the Penrose diagram as already stated in Subsection 2.2.

A set of coordinates (5.3.24) are connected with the other coordinates (5.3.23) by the relations,

$$t_{\text{in}} \to t_{\text{out}} - i\frac{\pi}{2\kappa}, \quad (r_*)_{\text{in}} \to (r_*)_{\text{out}} + i\frac{\pi}{2\kappa}, \tag{5.3.25}$$

so that, with this mapping, $T_{\text{in}} \to T_{\text{out}}$ and $R_{\text{in}} \to R_{\text{out}}$ smoothly. Following the definition (5.3.18), we obtain the relations connecting the null coordinates defined inside and outside the horizon,

$$u_{\text{in}} \equiv t_{\text{in}} - (r_*)_{\text{in}} \to u_{\text{out}} - i\frac{\pi}{\kappa}, \tag{5.3.26}$$

$$v_{\text{in}} \equiv t_{\text{in}} + (r_*)_{\text{in}} \to v_{\text{out}}. \tag{5.3.27}$$

This mapping is not defined if these coordinates are restricted to be real numbers. However, we can define it by extending these coordinates to complex numbers as in (5.3.25) and (5.3.26). We regard the appearance of the complex coordinates as a manifestation of quantum tunneling. In analogy to the quantum tunneling in ordinary quantum mechanics, our picture is that the inside solutions (5.3.19) and the outside solutions (5.3.20) of the infinitely high but very thin barrier, which is located on top of the horizon, are connected via the complex coordinates (i.e., tunneling); the precise connection of two coordinates is determined by asking the smooth connection of the Kruskal-Szekeres coordinates at the horizon. Under these transformations the inside and outside modes are connected by,

$$\phi^R_{\text{in}} \equiv \exp\left(-\frac{i}{\hbar}\omega u_{\text{in}}\right) \to \exp\left(-\frac{\pi\omega}{\hbar\kappa}\right)\phi^R_{\text{out}}, \tag{5.3.28}$$

$$\phi^L_{\text{in}} \equiv \exp\left(-\frac{i}{\hbar}\omega v_{\text{in}}\right) \to \phi^L_{\text{out}}, \tag{5.3.29}$$

where $\exp\left(-\frac{\pi\omega}{\hbar\kappa}\right)$ of (5.3.28) stands for the effect of the tunneling mechanism. We would like to note that these scalar fields are still identified with Schrödinger amplitude for a single particle state. Therefore, the squared absolute value of the wave function represents the probability. The

probability of the tunneling effect is given by

$$\Gamma = \exp\left(-\frac{2\pi\omega}{\hbar\kappa}\right). \tag{5.3.30}$$

We find that this result in the natural system of units $\hbar = 1$ agrees with Parikh and Wilczek's result as in (5.1.18).

To find the black body spectrum and the Hawking flux, Banerjee and Majhi considered $n$ number of non-interacting virtual pairs that are created inside the black hole. According to quantum field theory, a wave function of one-particle system $\phi$ is related to the second-quantized field operator $\hat{\psi}$ by

$$\phi = \langle 0|\hat{\psi}|\omega\rangle. \tag{5.3.31}$$

By following the approach of Banerjee and Majhi [8], each of these pairs is represented by the modes defined in the first set of (5.3.19) and (5.3.20). Then they defined the physical state of the system, observed from outside, as

$$|\Psi\rangle = N\sum_n |n^L_{\text{in}}\rangle \otimes |n^R_{\text{in}}\rangle, \tag{5.3.32}$$

where $|n^{L(R)}_{\text{in}}\rangle$ is the number state of left (right) going modes inside the black hole and $N$ is a normalization constant. From the transformations of both (5.3.28) and (5.3.29), we obtain

$$|\Psi\rangle = N\sum_n e^{-\frac{\pi n\omega}{\hbar\kappa}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle. \tag{5.3.33}$$

Here $N$ can be determined by using the normalization condition $\langle\Psi|\Psi\rangle = 1$. It is natural to determine the normalization constant $N$ for the state outside the black hole, because the observer exists outside the black hole. Thus we obtain

$$N = \frac{1}{\left(\sum_n e^{-\frac{2\pi n\omega}{\hbar\kappa}}\right)^{\frac{1}{2}}}. \tag{5.3.34}$$

For bosons ($n = 0, 1, 2, \cdots$), $N_{(\text{boson})}$ is calculated as

$$N_{(\text{boson})} = \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right)^{\frac{1}{2}}. \tag{5.3.35}$$

For fermions ($n = 0, 1$), $N_{(\text{fermion})}$ is similarly calculated as

$$N_{(\text{fermion})} = \left(1 + e^{-\frac{2\pi\omega}{\hbar\kappa}}\right)^{-\frac{1}{2}}. \tag{5.3.36}$$

By substituting (5.3.35) or (5.3.36) into (5.3.33), we obtain the normalized physical states of a system of bosons or fermions

$$|\Psi\rangle_{(\text{boson})} = \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right)^{\frac{1}{2}} \sum_n e^{-\frac{\pi n\omega}{\hbar\kappa}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle, \tag{5.3.37}$$

$$|\Psi\rangle_{(\text{fermion})} = \left(1 + e^{-\frac{2\pi\omega}{\hbar\kappa}}\right)^{-\frac{1}{2}} \sum_n e^{-\frac{\pi n\omega}{\hbar\kappa}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle. \tag{5.3.38}$$

From here on this analysis will be only for bosons since the analysis for fermions is identical. For bosons, the density matrix operator of the system is given by

$$\begin{aligned}\hat{\rho}_{(\text{boson})} &\equiv |\Psi\rangle_{(\text{boson})}\langle\Psi|_{(\text{boson})} \\ &= \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \sum_{n,m} e^{-\frac{\pi\omega}{\hbar\kappa}(n+m)} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle\langle m^R_{\text{out}}| \otimes \langle m^L_{\text{out}}|. \end{aligned} \tag{5.3.39}$$

By tracing out the left going modes, we obtain the reduced density matrix for the right going modes,

$$\hat{\rho}^{(R)}_{(\text{boson})} = \text{Tr}\left(\hat{\rho}^{(R)}_{(\text{boson})}\right) \tag{5.3.40}$$

$$= \sum_n \langle n^L_{\text{out}}|\hat{\rho}_{(\text{boson})}|n^L_{\text{out}}\rangle \tag{5.3.41}$$

$$= \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \sum_n e^{-\frac{2\pi n\omega}{\hbar\kappa}} |n^R_{\text{out}}\rangle\langle n^R_{\text{out}}|. \tag{5.3.42}$$

Therefore the average number of particles detected at asymptotic infinity is given by

$$\langle n\rangle_{\text{boson}} = \text{Tr}\left(\hat{n}\hat{\rho}^{(R)}_{(\text{boson})}\right) \tag{5.3.43}$$

$$= \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \sum_n n e^{-\frac{2\pi n\omega}{\hbar\kappa}} \tag{5.3.44}$$

$$= \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \left(-\frac{\hbar\kappa}{2\pi}\right) \frac{\partial}{\partial\omega} \left(\sum_{n=0}^{\infty} e^{-\frac{2\pi n\omega}{\hbar\kappa}}\right) \tag{5.3.45}$$

$$= \left(1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}\right) \left(-\frac{\hbar\kappa}{2\pi}\right) \frac{\partial}{\partial\omega} \left(\frac{1}{1 - e^{-\frac{2\pi\omega}{\hbar\kappa}}}\right) \tag{5.3.46}$$

$$= \frac{1}{e^{\frac{2\pi\omega}{\hbar\kappa}} - 1}, \tag{5.3.47}$$

where the trace is taken over all $|n^{(R)}_{\text{out}}\rangle$ eigenstates. This result in the natural system of units ($\hbar = 1$) agrees with the result of Hawking's original derivation as in (3.1.70). Similar analysis for fermions leads to the Fermi distribution given by

$$\langle n\rangle_{\text{fermion}} = \frac{1}{e^{\frac{2\pi\omega}{\hbar\kappa}} + 1}. \tag{5.3.48}$$

Correspondingly, the Hawking flux $F_{\rm H}$ can be obtained by integrating the above distribution functions over all $\omega$'s. By following the discussion in Subsection 5.4.3, we would like to evaluate the Hawking flux for fermions here. It is given by

$$F_{\rm H} = \frac{1}{\pi}\int_0^\infty \frac{\omega}{e^{\frac{2\pi\omega}{\hbar\kappa}}+1}d\omega = \frac{\hbar^2\kappa^2}{48\pi} = \frac{\pi}{12\beta^2}, \tag{5.3.49}$$

where we used

$$\beta \equiv \frac{1}{\mathcal{T}_{BH}} \equiv \frac{2\pi}{\hbar\kappa}. \tag{5.3.50}$$

This result agrees with the case of $a = 0$ in (4.3.32), i.e., the second term of (4.3.32).

Thus Banerjee and Majhi directly showed how to reproduce the black body spectrum and the Hawking flux with the Hawking temperature from the expectation value of the number operator by using the properties of the tunneling mechanism. Therefore, it is shown that black holes are not merely the thermal body but the black body in the derivation from the tunneling mechanism. In this sense, it may be said that the derivation on the basis of the tunneling mechanism became more satisfactory.

## 5.4 Hawking Radiation from Kerr-Newman Black Hole and Tunneling Mechanism

A variant of the approach to the derivation of Hawking radiation from the tunneling mechanism was suggested by Banerjee and Majhi [8] as described in the preceding section. However, as stated in their paper, their result is valid only for black holes with spherically symmetric geometry such as Schwarzschild or Reissner-Nordström black holes in the 4-dimensional theory. According to the black hole uniqueness theorem (see Section 2.1), 4-dimensional black holes have not only a mass and a charge but also angular momentum. In 4 dimensions, the Kerr-Newman black hole, which has both the charge and angular momentum, is the most general black hole and its geometry becomes spherically asymmetric because of its own rotation. There exist several previous works on a rotating black hole in the framework of the tunneling method (see for example [103–107]), but they are mathematically very involved.

In this section, we would like to extend the method of Banerjee and Majhi based the tunneling mechanism to the case of the Kerr-Newman black hole [9]. As shown in Section 2.4, we recall that the 4-dimensional Kerr-Newman metric effectively becomes a 2-dimensional spherically symmetric metric by using the technique of the dimensional reduction near the horizon. To the best of

my knowledge, there is no derivation of the spectrum by using the technique of the dimensional reduction in the tunneling mechanism. Therefore, we believe that this derivation clarifies some aspects of the tunneling mechanism. The essential idea is as follows: We consider the action for a scalar field. We can then ignore the mass, potential and interaction terms in the action because the kinetic term dominates in the high-energy theory near the horizon. By expanding the scalar field in terms of the spherical harmonics and using properties at horizon, we find that the integrand in the action dose not depend on angular variables. Thus we find that the 4-dimensional action with the Kerr-Newman metric effectively becomes a 2-dimensional action with the spherically symmetric metric.

We note that this technique is valid only for the region near the horizon. The use of the above technique in the tunneling mechanism is justified since the tunneling effect is also the quantum effect arising within the Planck length near the horizon region. By this procedure, the metric for the Kerr-Newman black hole becomes an effectively 2-dimensional spherically symmetric metric, and we can use the approach of Banerjee and Majhi which is valid for black holes with spherically symmetric geometry. We can thus derive the black body spectrum and Hawking flux for the Kerr-Newman black hole in the tunneling mechanism. We would like to suggest that the technique of the dimensional reduction is also valid for Parikh and Wilczek's original method in the tunneling mechanism.

The contents of this section are as follows. In Subsection 5.4.1, we show how to define the Kruskal-like coordinate for the effective 2-dimensional metric. In Subsection 5.4.2, we discuss the tunneling mechanism for the case of a Kerr-Newman black hole. In Subsection 5.4.3, we show the black body spectrum and the Hawking flux for the case of the Kerr-Newman black hole.

### 5.4.1 Kruskal-like coordinates for the effective 2-dimensional metric

In this subsection, we briefly explain that the 4-dimensional Kerr-Newman metric becomes a 2-dimensional spherically symmetric metric by using technique of the dimensional reduction near the horizon. Then the Kruskal-like coordinates for the reduced metric are required instead of the Kruskal-Szekeres coordinates for the 2-dimensional Schwarzschild metric as in (5.3.21) and (5.3.22). We would like to show how to obtain the Kruskal-like coordinates from the reduced metric.

For a rotating and charged black hole, the metric is given by the Kerr-Newman metric (2.1.8)

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta}{\Sigma}(r^2+a^2-\Delta)dtd\varphi \\ &-\frac{a^2\Delta\sin^2\theta-(r^2+a^2)^2}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{5.4.1}$$

where notations were respectively defined in Section 2.1. It follows from this expression that the Kerr-Newman metric is spherically asymmetric geometry.

In the Kerr-Newman black hole background, the 4-dimensional action for a complex scalar field is given by (2.4.1)

$$S = \int d^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu + ieV_\mu)\phi^*(\partial_\nu - ieV_\nu)\phi + S_{\text{int}}, \tag{5.4.2}$$

where the first term is the kinetic term, the second term $S_{\text{int}}$ represents the mass, potential and interaction terms and $V_\mu$ is a gauge field associated with the Coulomb potential of the black hole.

By using the technique of the dimensional reduction near the horizon, it can be shown that the 4-dimensional action (5.4.2) becomes

$$S_{(\text{H})} = -\sum_{l,m}\int dtdr\Phi\phi^*_{lm}\left[g^{tt}\left(\partial_t - iA_t\right)^2 + \partial_r g^{rr}\partial_r\right]\phi_{lm}, \tag{5.4.3}$$

as shown in (2.4.22). Then the effective metric near the horizon is given by (2.4.23)

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{5.4.4}$$

where $f(r)$ is defined by (2.4.10)

$$f(r) \equiv \frac{\Delta}{r^2+a^2}. \tag{5.4.5}$$

We note that this function $f(r)$ certainly contains the effect of the rotating black hole expressed by $a$. From the form of (5.4.4), we find that a 4-dimensional Kerr-Newman metric (5.4.1) effectively becomes the 2-dimensional spherically symmetric metric near the horizon. This expression also shows that it is reasonable to consider only the $(r-t)$ sector of the 4-dimensional metric and massless particles without interactions, which were used in previous works [7,8].

Banerjee and Majhi used Kruskal-Szekeres coordinates for a spherically symmetric metric such as Schwarzschild or Reissner-Nordström metrics as in (5.3.23) and (5.3.24). In general, the concrete forms of Kruskal-Szekeres coordinates for Schwarzschild or Reissner-Nordström metrics are well known (as for the case of a Reissner-Nordström metric, for example, see §3.1 in [23]). In order to

extend Banerjee and Majhi's method to the case of a Kerr-Newman black hole, we need to derive the Kruskal-like coordinates for the effective reduced metric (5.4.4) following the derivation of the Kruskal-Szekeres coordinates in the standard textbook. In this derivation, we apply a series of coordinate transformations. Note that all the variables which appear in our transformations are defined to be real numbers.

Now the metric is given by (5.4.4). By using (2.1.18), $f(r)$ is also written as

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 + a^2}, \tag{5.4.6}$$

and $r_{+(-)}$ is the outer (inner) horizon given by (2.1.17).

As a first step of coordinate transformations, we use the tortoise coordinate defined by (2.4.10)

$$dr_* \equiv \frac{1}{f(r)} dr. \tag{5.4.7}$$

The metric (5.4.4) is then written by

$$ds^2 = -f(r)(dt - dr_*)(dt + dr_*). \tag{5.4.8}$$

By integrating (5.4.7) over $r$ from 0 to r, we obtain

$$r_* = r + \frac{1}{2\kappa_+} \ln \frac{|r - r_+|}{r_+} + \frac{1}{2\kappa_-} \ln \frac{|r - r_-|}{r_-} + C, \tag{5.4.9}$$

where $\kappa_{+(-)}$ is the surface gravity on the outer (inner) horizon and $C$ is generally a pure imaginary integration constant which appears in the analytic continuation. However, as already stated, we would like to treat the case where all the variables (or parameters) are defined in the range of real numbers. Thus we need to consider the three cases $r > r_+$, $r_+ < r < r_-$ and $r_- < r < 0$ with respect to the range of $r$. Actually, we have only to consider the two cases $r > r_+$ and $r_+ < r < r_-$ because of the consideration near the outer horizon. When $r > r_+$, the relation (5.4.9) becomes

$$r_* = r + \frac{1}{2\kappa_+} \ln \frac{r - r_+}{r_+} + \frac{1}{2\kappa_-} \ln \frac{r - r_-}{r_-}. \tag{5.4.10}$$

As the second step we use the retarded time $u$ and the advanced time $v$ defined by

$$\left.\begin{aligned} u &\equiv t - r_* = t - r - \frac{1}{2\kappa_+} \ln \frac{r - r_+}{r_+} - \frac{1}{2\kappa_-} \ln \frac{r - r_-}{r_-}, \\ v &\equiv t + r_* = t + r + \frac{1}{2\kappa_+} \ln \frac{r - r_+}{r_+} + \frac{1}{2\kappa_-} \ln \frac{r - r_-}{r_-}. \end{aligned}\right\} \quad \text{when} \quad r > r_+. \tag{5.4.11}$$

The metric (5.4.8) is then written as

$$ds^2 = -f(r) du dv. \tag{5.4.12}$$

As the third step we use the following coordinate transformations $U$ and $V$ defined by

$$\left.\begin{aligned} U &\equiv -e^{-\kappa_+ u} = -\left(\frac{r-r_+}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r} e^{-\kappa_+ t}, \\ V &\equiv e^{\kappa_+ v} = \left(\frac{r-r_+}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r} e^{\kappa_+ t}. \end{aligned}\right\} \quad \text{when} \quad r > r_+. \tag{5.4.13}$$

The metric (5.4.12) is then written as

$$ds^2 = -\frac{r_+ r_-}{\kappa_+^2}\frac{e^{-2\kappa_+ r}}{r^2+a^2}\left(\frac{r_-}{r-r_-}\right)^{\frac{\kappa_+}{\kappa_-}-1} dU dV. \tag{5.4.14}$$

As the final step we use the following coordinate transformations $T$, $R$ defined by

$$\left.\begin{aligned} T &\equiv \frac{1}{2}(V+U) = \left(\frac{r-r_+}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r}\sinh(\kappa_+ t), \\ R &\equiv \frac{1}{2}(V-U) = \left(\frac{r-r_+}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r}\cosh(\kappa_+ t). \end{aligned}\right\} \quad \text{when} \quad r > r_+. \tag{5.4.15}$$

The metric (5.4.14) is then written as

$$ds^2 = \frac{r_+ r_-}{\kappa_+^2}\frac{e^{-2\kappa_+ r}}{r^2+a^2}\left(\frac{r_-}{r-r_-}\right)^{\frac{\kappa_+}{\kappa_-}-1}(-dT^2+dR^2). \tag{5.4.16}$$

Similarly, we consider the case of $r_+ > r > r_-$. When $r_+ > r > r_-$, the relation (5.4.9) becomes

$$r_* = r + \frac{1}{2\kappa_+}\ln\frac{r_+-r}{r_+} + \frac{1}{2\kappa_-}\ln\frac{r-r_-}{r_-}. \tag{5.4.17}$$

As for the remaining coordinate transformations, we use the following ones

$$\left.\begin{aligned} u &\equiv t - r_* = t - r - \frac{1}{2\kappa_+}\ln\frac{r_+-r}{r_+} - \frac{1}{2\kappa_-}\ln\frac{r-r_-}{r_-}, \\ v &\equiv t + r_* = t + r + \frac{1}{2\kappa_+}\ln\frac{r_+-r}{r_+} + \frac{1}{2\kappa_-}\ln\frac{r-r_-}{r_-}, \end{aligned}\right\} \quad \text{when} \quad r_+ > r > r_-, \tag{5.4.18}$$

$$\left.\begin{aligned} U &\equiv e^{-\kappa_+ u} = \left(\frac{r_+-r}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r} e^{-\kappa_+ t}, \\ V &\equiv e^{\kappa_+ v} = \left(\frac{r_+-r}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r} e^{\kappa_+ t}, \end{aligned}\right\} \quad \text{when} \quad r_+ > r > r_-, \tag{5.4.19}$$

$$\left.\begin{aligned} T &\equiv \frac{1}{2}(V+U) = \left(\frac{r_+-r}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r}\cosh(\kappa_+ t), \\ R &\equiv \frac{1}{2}(V-U) = \left(\frac{r_+-r}{r_+}\right)^{\frac{1}{2}}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{2\kappa_-}} e^{\kappa_+ r}\sinh(\kappa_+ t). \end{aligned}\right\} \quad \text{when} \quad r_+ > r > r_-. \tag{5.4.20}$$

Of course, by performing these coordinate transformations, the corresponding metrics (5.4.12), (5.4.14) and (5.4.16) do not change. The two sets of coordinates introduced here, both of (5.4.15)

and (5.4.20), are in fact the Kruskal-like coordinates. In the Schwarzshild case ($a = Q = 0$), (5.4.15) and (5.4.20) respectively agree with (5.3.21) and (5.3.22). Finally, by rewriting the expressions written in terms of $r$ to the ones in terms of $r_*$ in the formulas (5.4.15) and (5.4.20), we obtain

$$\left.\begin{array}{l} T = \exp\left[\kappa_+ r_*\right] \sinh\left(\kappa_+ t\right) \\ R = \exp\left[\kappa_+ r_*\right] \cosh\left(\kappa_+ t\right) \end{array}\right\} \quad \text{when} \quad r > r_+, \tag{5.4.21}$$

$$\left.\begin{array}{l} T = \exp\left[\kappa_+ r_*\right] \cosh\left(\kappa_+ t\right) \\ R = \exp\left[\kappa_+ r_*\right] \sinh\left(\kappa_+ t\right) \end{array}\right\} \quad \text{when} \quad r_+ > r > r_-. \tag{5.4.22}$$

These results agree with (5.3.23) and (5.3.24). Thus we can regard these coordinate variables as the Kruskal-like coordinate variables for the effective reduced metric.

### 5.4.2 Tunneling mechanism

In this subsection, we discuss the connection between states inside and outside the black hole to analyze the tunneling effect in the induced metric. We consider the Klein-Gordon equation near the horizon. In Section 2.4, we showed that we can regard the 4-dimensional Kerr metric as the 2-dimensional spherically symmetric metric in the region near the horizon. As already stated, since the kinetic term dominates in the high-energy theory near the horizon, we can ignore the mass, potential and interaction terms. We obtain the Klein-Gordon equation with the gauge field from the action (2.4.15)

$$\left[\frac{1}{f(r)}\left(\partial_t - iA_t\right)^2 - f(r)\partial_r^2 - f'(r)\partial_r\right]\phi = 0, \tag{5.4.23}$$

where $A_t$ is defined in (2.4.21) and $f(r)$ is defined in (5.4.6). Of course, this equation can be obtained from the general Klein-Gordon equation for a free particle with the gauge field in 2-dimensional space-time

$$g^{\mu\nu}(\nabla_\mu - iA_\mu)(\nabla_\nu - iA_\nu)\phi = 0, \tag{5.4.24}$$

where $\nabla_\mu$ is the covariant derivative as in (3.1.7). In a manner similar to the procedure explained in Section 5.3, we adopt the standard WKB ansatz

$$\phi(r,t) = e^{\frac{i}{\hbar}S(r,t)}, \tag{5.4.25}$$

and substituting the expansion of $S(r,t)$

$$S(r,t) = S_0(r,t) + \sum_{i=1}^{\infty} \hbar^i S_i(r,t), \tag{5.4.26}$$

in (5.4.23), we obtain, in the semiclassical limit (i.e., keeping only $S_0$),

$$\partial_t S_0(r,t) = \pm f(r)\partial_r S_0(r,t). \tag{5.4.27}$$

We find that terms including the gauge field vanished in the semiclassical limit. This equation completely agrees with the equation in [8] although the content of $f(r)$ is different from that used in [8]. From the Hamilton-Jacobi equation as in (5.3.9) and (5.4.27), we obtain

$$S_0(r,t) = -\left(\omega - e\Phi_{\mathrm{H}} - m\Omega_{\mathrm{H}}\right)(t \pm r_*) \equiv -\omega'(t \pm r_*), \tag{5.4.28}$$

where $r_*$ is the tortoise coordinate defined by (2.4.10), $\omega$ is the characteristic frequency; $\Phi_{\mathrm{H}}$ is the electric potential defined by (2.1.25), and $\Omega_{\mathrm{H}}$ is the angular frequency on the horizon respectively defined by (2.1.24),

$$\Phi_{\mathrm{H}} = \frac{Qr_+}{r_+^2 + a^2}, \quad \Omega_{\mathrm{H}} = \frac{a}{r_+^2 + a^2}. \tag{5.4.29}$$

We also used the fact that the Hamiltonian is asymptotically given by

$$H = \omega - e\Phi_{\mathrm{H}} - m\Omega_{\mathrm{H}}. \tag{5.4.30}$$

Thus we obtain the semiclassical solution for the scalar field

$$\phi(r,t) = \exp\left[-\frac{i}{\hbar}\omega'(t \pm r_*)\right]. \tag{5.4.31}$$

If one considers the metric at all regions for the charged and rotating black hole, the exterior metric of the horizon is given by the 4-dimensional Kerr-Newman metric (5.4.1). However, we showed that the metric near the horizon can be regarded as the 2-dimensional spherically symmetric metric as in (5.4.4). We thus use the metric (5.4.4) in the region near the horizon. But we do not know the interior metric of the black hole. However, if the interior metric is smoothly connected to the exterior metric through the horizon, we may be able to identify the interior metric near the horizon to be the same as the exterior metric near the horizon. We thus suppose that the interior metric near the horizon is given by (5.4.4). Furthermore, since the tunneling effect is the quantum effect arising within the Planck length in the near horizon region, we have to consider both the inside and outside regions which are very close to the horizon.

Here we use both the retarded time $u$ and the advanced time $v$ defined by

$$u \equiv t - r_*, \qquad v \equiv t + r_*. \tag{5.4.32}$$

We can then separate the scalar field (5.4.31) into the ingoing (left handed) modes and outgoing (right handed) modes. In the regions $r_+ - \varepsilon < r < r_+$, and $r_+ \leq r < r_+ - \varepsilon$, respectively, we express the field $\phi$ as

$$\left.\begin{aligned} \phi^R_{\rm in} &= \exp\left(-\frac{i}{\hbar}\omega' u_{\rm in}\right) \\ \phi^L_{\rm in} &= \exp\left(-\frac{i}{\hbar}\omega' v_{\rm in}\right) \end{aligned}\right\} \qquad (r_+ - \varepsilon < r < r_+), \tag{5.4.33}$$

$$\left.\begin{aligned} \phi^R_{\rm out} &= \exp\left(-\frac{i}{\hbar}\omega' u_{\rm out}\right) \\ \phi^L_{\rm out} &= \exp\left(-\frac{i}{\hbar}\omega' v_{\rm out}\right) \end{aligned}\right\} \qquad (r_+ < r < r_+ + \varepsilon), \tag{5.4.34}$$

where $\varepsilon$ is an arbitrarily small constant, "R (L)" stands for the right (left) modes and "in (out)" stands for the inside (outside) of the black hole, respectively (see Fig. 5.4). Here we note that these fields are defined both the inside and outside regions which are very close to the horizon. As for the definition of the fields in the region close to the horizon, there are related discussions in the literatures [113, 114].

As shown in Subsection 5.4.1, we use the Kruskal-like coordinate variables

$$\left.\begin{aligned} T_{\rm out} &= \exp\left[\kappa_+ (r_*)_{\rm out}\right] \sinh\left(\kappa_+ t_{\rm out}\right) \\ R_{\rm out} &= \exp\left[\kappa_+ (r_*)_{\rm out}\right] \cosh\left(\kappa_+ t_{\rm out}\right) \end{aligned}\right\}. \tag{5.4.35}$$

$$\left.\begin{aligned} T_{\rm in} &= \exp\left[\kappa_+ (r_*)_{\rm in}\right] \cosh\left(\kappa_+ t_{\rm in}\right) \\ R_{\rm in} &= \exp\left[\kappa_+ (r_*)_{\rm in}\right] \sinh\left(\kappa_+ t_{\rm in}\right) \end{aligned}\right\}, \tag{5.4.36}$$

Both of the relations (5.4.35) and (5.4.36) agree with the relations in [8].

In general, the Schwarzschild and Kerr-Newman metrics describe the behavior outside the black hole. Consequently, the readers may wonder if these metrics defined by the Kruskal coordinates can really describe the behavior inside the black hole. In our case, however, we study the tunneling effect across the horizon and thus we study the behavior in the very small regions near the horizon. In such analysis, due to the reasons of continuity, it may not be unnatural to assume that the Kruskal coordinates (5.4.35) and (5.4.36) can be used in both of outside and inside regions of near the horizon.

A set of coordinates (5.4.35) are connected with the other coordinates (5.4.36) by the relations,

$$t_{\rm in} \to t_{\rm out} - i\frac{\pi}{2\kappa_+}, \quad (r_*)_{\rm in} \to (r_*)_{\rm out} + i\frac{\pi}{2\kappa_+}, \tag{5.4.37}$$

so that, with this mapping, $T_{\rm in} \to T_{\rm out}$ and $R_{\rm in} \to R_{\rm out}$ smoothly. Following the definition (5.4.32),

we obtain the relations connecting the null coordinates defined inside and outside the horizon,

$$u_{\rm in} \equiv t_{\rm in} - (r_*)_{\rm in} \to u_{\rm out} - i\frac{\pi}{\kappa_+}, \tag{5.4.38}$$

$$v_{\rm in} \equiv t_{\rm in} + (r_*)_{\rm in} \to v_{\rm out}. \tag{5.4.39}$$

This mapping is not defined if these coordinates are restricted to be real numbers. However, we can define it by extending these coordinates to complex numbers as in (5.4.37) and (5.4.38). We regard the appearance of the complex coordinates as a manifestation of quantum tunneling. In analogy to the quantum tunneling in ordinary quantum mechanics, our picture is that the inside solutions (5.4.33) and the outside solutions (5.4.34) of the infinitely high but very thin barrier, which is located on top of the horizon, are connected via the complex coordinates (i.e., tunneling); the precise connection of two coordinates is determined by asking the smooth connection of the Kruskal-like coordinates at the horizon. Under these transformations the inside and outside modes are connected by,

$$\phi^R_{\rm in} \equiv \exp\left(-\frac{i}{\hbar}\omega' u_{\rm in}\right) \to \exp\left(-\frac{\pi\omega'}{\hbar\kappa_+}\right)\phi^R_{\rm out}, \tag{5.4.40}$$

$$\phi^L_{\rm in} \equiv \exp\left(-\frac{i}{\hbar}\omega' v_{\rm in}\right) \to \phi^L_{\rm out}. \tag{5.4.41}$$

As already discussed by Banerjee and Majhi in [8], the essential idea of the tunneling mechanism is that a particle-antiparticle pair is formed close to the horizon. This pair creation may arise inside the black hole ( in the region close to the horizon ), since the space-time is locally flat. The ingoing mode is trapped inside the horizon while the outgoing mode can quantum mechanically tunnel through the horizon. The outgoing mode is then observed at infinity as the Hawking flux. We find that the effect of the ingoing mode inside the horizon do not appear outside the horizon as in (5.4.41) since $v_{\rm in}$ changes to $v_{\rm out}$ without an extra term under the transformation connecting the null coordinates defined inside and outside the horizon as in (5.4.39). On the other hand, we find that the effect of the outgoing mode inside the horizon appear with a non-negligible probability by tunneling through the horizon quantum mechanically as in (5.4.40). This consideration agrees with the concept of tunneling mechanism. Furthermore, we showed that we can treat the Kerr-Newman metric as a 2-dimensional spherically symmetric metric with a 2-dimensional effective gauge field just as in the case of the simplest Schwarzschild metric in the tunneling mechanism.

### 5.4.3 Black body spectrum and Hawking flux

In this subsection, we show how to derive the Hawking black body spectrum for a Kerr-Newman black hole by following the approach of Banerjee and Majhi as shown in Section 5.3. First, we consider $n$ number of non-interacting virtual pairs that are created inside the black hole. Then the physical state of the system is conventionally written as

$$|\Psi\rangle = N \sum_n |n^L_{\text{in}}\rangle \otimes |n^R_{\text{in}}\rangle, \tag{5.4.42}$$

where $|n^{L(R)}_{\text{in}}\rangle$ is the number state of left (right) going modes inside the black hole and $N$ is a normalization constant. From the transformations of both (5.4.40) and (5.4.41), we obtain

$$|\Psi\rangle = N \sum_n e^{-\frac{\pi n \omega'}{\hbar\kappa_+}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle. \tag{5.4.43}$$

Here $N$ can be determined by using the normalization condition $\langle\Psi|\Psi\rangle = 1$. It is natural to determine the normalization constant $N$ for the state outside the black hole, because the observer exists outside the black hole. Thus we obtain

$$N = \frac{1}{\left(\sum_n e^{-\frac{2\pi n\omega'}{\hbar\kappa_+}}\right)^{\frac{1}{2}}}. \tag{5.4.44}$$

For bosons ($n = 0, 1, 2, \cdots$), $N_{(\text{boson})}$ is calculated as

$$N_{(\text{boson})} = \left(1 - e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)^{\frac{1}{2}}. \tag{5.4.45}$$

For fermions ($n = 0, 1, 2$), $N_{(\text{fermion})}$ is also calculated as

$$N_{(\text{fermion})} = \left(1 + e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)^{-\frac{1}{2}}. \tag{5.4.46}$$

By substituting (5.4.45) or (5.4.46) into (5.4.43), we obtain the normalized physical states of a system of bosons or fermions

$$|\Psi\rangle_{(\text{boson})} = \left(1 - e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)^{\frac{1}{2}} \sum_n e^{-\frac{\pi n\omega'}{\hbar\kappa_+}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle, \tag{5.4.47}$$

$$|\Psi\rangle_{(\text{fermion})} = \left(1 + e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)^{-\frac{1}{2}} \sum_n e^{-\frac{\pi n\omega'}{\hbar\kappa_+}} |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle. \tag{5.4.48}$$

Similarly to Section 5.3, we will present the analysis for bosons only. The density matrix operator of the system is given by

$$\begin{aligned}\hat{\rho}_{(\text{boson})} \equiv& |\Psi\rangle_{(\text{boson})}\langle\Psi|_{(\text{boson})} \\ =& \left(1 - e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)\sum_{n,m} e^{-\frac{\pi\omega'}{\hbar\kappa_+}(n+m)} \\ & \times |n^L_{\text{out}}\rangle \otimes |n^R_{\text{out}}\rangle\langle m^R_{\text{out}}| \otimes \langle m^L_{\text{out}}|. \end{aligned} \tag{5.4.49}$$

By tracing out the left going modes, we obtain the reduced density matrix for the right going modes,

$$\hat{\rho}^{(R)}_{(\text{boson})} = \text{Tr}\left(\hat{\rho}^{(R)}_{(\text{boson})}\right) \tag{5.4.50}$$

$$= \sum_n \langle n^L_{\text{out}}|\hat{\rho}_{(\text{boson})}|n^L_{\text{out}}\rangle \tag{5.4.51}$$

$$= \left(1 - e^{-\frac{2\pi\omega'}{\hbar\kappa_+}}\right)\sum_n e^{-\frac{2\pi n\omega'}{\hbar\kappa_+}}|n^R_{\text{out}}\rangle\langle n^R_{\text{out}}|. \tag{5.4.52}$$

Then, the expectation value of the number operator $\hat{n}$ is given by

$$\langle n\rangle_{\text{boson}} = \text{Tr}\left(\hat{n}\hat{\rho}^{(R)}_{(\text{boson})}\right) \tag{5.4.53}$$

$$= \frac{1}{e^{\frac{2\pi\omega'}{\hbar\kappa_+}} - 1} \tag{5.4.54}$$

$$= \frac{1}{e^{\beta(\omega - e\Phi - m\Omega)} - 1} \tag{5.4.55}$$

where in the last line we used the definition (5.4.28) and we identify the Hawking temperature $\mathcal{T}_{\text{BH}}$ by

$$\beta \equiv \frac{1}{\mathcal{T}_{\text{BH}}} \equiv \frac{2\pi}{\hbar\kappa_+}. \tag{5.4.56}$$

This result corresponds to the black body spectrum with the Hawking temperature and agrees with previous works in the Kerr-Newman black hole background [2]. Similar analysis for fermions leads to the Fermi distribution

$$\langle n\rangle_{\text{fermion}} = \frac{1}{e^{\beta(\omega - e\Phi - m\Omega)} + 1}. \tag{5.4.57}$$

Moreover, the Hawking flux $F_H$ is derived by integrating the sum of the distribution function for a particle with a quantum number $(e, m)$ and its antiparticle with $(-e, -m)$ over all $\omega$'s. However, as shown in Section 2.3, boson fields display the superradiance provided that they have frequency in a

certain range whereas fermion fields do not. We therefore evaluate the Hawking flux for fermions. It is given by

$$F_{\mathrm{H}} = \int_0^\infty \frac{d\omega}{2\pi}\omega\left[\frac{1}{e^{\beta(\omega-e\Phi-m\Omega)}+1}+\frac{1}{e^{\beta(\omega+e\Phi+m\Omega)}+1}\right] \tag{5.4.58}$$

$$= \frac{\pi}{12\beta^2}+\frac{1}{4\pi}(e\Phi+m\Omega)^2. \tag{5.4.59}$$

This result agrees with the previous result [5] (see Appendix A in [5]).

One might be surprised by the sudden appearance of fermions. However, we can explicitly present an answer to the question. As shown in Section 2.4, we know that the effective theory near the horizon becomes a 2-dimensional theory. According to 2-dimensional quantum field theory, it is known that there exists the boson-fermion duality [115]. Namely, the 2-dimensional boson theory can be treated as the 2-dimensional fermion theory by the fermionization. We can also discuss the tunneling mechanism for fermions in a manner similar to bosons.

Before closing, we discuss the black hole entropy. Since a particle emitted by the black hole has the Hawking temperature, it is natural to consider that the black hole itself has the same temperature. Thus we can obtain the black hole entropy $dS_{\mathrm{BH}}$ from a thermodynamic consideration

$$dS_{\mathrm{BH}} = \left(\frac{dM}{\mathcal{T}_{BH}}\right). \tag{5.4.60}$$

By integrating Eq. (5.4.60), the entropy agrees with the Bekenstein-Hawking entropy

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4\hbar}, \tag{5.4.61}$$

where $A_{\mathrm{BH}}$ is the surface area of the black hole

$$A_{\mathrm{BH}} = 4\pi(r_+^2+a^2). \tag{5.4.62}$$

This result agrees with the Bekenstein-Hawking entropy (5.4.61). On the other hand, we defined quantum states as in (5.4.42) and obtained the reduced density matrix as in (5.4.52). We can evaluate the entropy for these states by using the von Neumann entropy formula. However, we must mention that it is not the entropy of the black hole itself but rather the entropy of the boson field. Our method does not allow us to derive the entropy for a black hole with a finite temperature by counting the number of quantum states associated with the black hole. We thus simply derived the black hole entropy from thermodynamic considerations as in (5.4.60) following previous works. The derivation of the black hole entropy by counting the number of black hole quantum states remains as one of future problems.

Finally, we mention that further recent developments associated with this derivation are given in [116–118].

# Chapter 6

# Discussion and Conclusion

In this thesis, we investigated the black hole radiation which is commonly called Hawking radiation. Hawking radiation is one of very interesting phenomena where both of general relativity and quantum theory play a role at the same time since Hawking radiation is derived by taking into account the quantum effects in the framework of general relativity. Hawking radiation is widely accepted by now because the same result is derived by several different methods. At the same time, there remain several aspects which have yet to be clarified. We attempted to clarify some arguments in previous works and present more satisfactory derivations of Hawking radiation.

In Chapter 2, we reviewed basic facts and various properties of black holes as the necessary preparation to discuss Hawking radiation. We showed that both of the black hole solutions and their types are given as a result of general relativity, and that Penrose diagrams are useful to understand the global structure of black hole space-time; a part of energy can be extracted from a rotating black hole by the Penrose process and the technique of the dimensional reduction plays an important role to understand the behavior of matter fields near the event horizon. We also discussed analogies between black hole physics and thermodynamics, and we explained a method to derive the black hole entropy which was suggested by Bekenstein. In particular, to understand the properties of black holes, it is very useful to consider black hole physics in terms of well-known thermodynamics. However, the corresponding relationships are no more than analogies in classical theory. If we would like to show the complete correspondence between black hole physics and thermodynamics, namely, to show that black holes actually have entropy and temperature, we need to explain black hole radiation. Although Bekenstein suggested that black holes can have entropy, the complete corresponding relationships was not established because Bekenstein was not able to explain the mechanism of black hole radiation.

In Chapter 3, we discussed several previous works on Hawking radiation. Hawking radiation which is derived by using quantum effects in black hole physics. By following Hawking's original derivation, we calculated the expectation value of the particle number by using the Bogoliubov transformations. As is well known, the result agrees with the black body spectrum with a certain temperature. By defining the temperature as the black hole temperature, it is confirmed that a black hole behaves as a black body and we can thus explain the black hole radiation. The existence of black hole radiation implies that the Hawking area theorem is violated. However, the second law of black hole physics holds in a suitably generalized form. From these considerations, we can understand the complete corresponding relationships between black hole physics and thermodynamics, and we find that the radiation-dominated tiny black hole will eventually evaporate at some point. We also briefly reviewed some representative derivations of Hawking radiation and stated that there remain some aspects to be clarified.

In Chapter 4, we discussed the derivation of Hawking radiation based on anomalies. The essential idea is as follows: We can divide the field associated with a particle into ingoing modes falling toward the horizon and outgoing modes moving away from the horizon near the horizon. Since none of ingoing modes at horizon are expected to affect the classical physics outside the horizon, we ignore the ingoing modes. Anomalies then appear since the effective theory is chiral near the horizon. Anomalies mean the symmetry breaking by the quantization, i.e., the corresponding conservation law is violated, and it is known that a source of energy is generated. We can consider that this source of energy corresponds to the characteristic energy flux of Hawking radiation. Since both of the anomalies and Hawking radiation are typical quantum effects, it is natural that Hawking radiation is related to anomalies. A method of deriving Hawking radiation based on the consideration of anomalies was first suggested by Robinson and Wilczek and generalized by Iso, Umetsu and Wilczek. However, there remained some aspects to be clarified. We clarified some arguments in previous works on this approach and presented a simple derivation of Hawking radiation from anomalies. We showed how to derive the Hawking flux by using the Ward identities for covariant currents and two physically-meaningful boundary conditions. This derivation has a merit that we can potentially incorporate matter fields with mass and interactions away from the horizon.

In Chapter 5, we discussed the derivation of Hawking radiation on the basis of quantum tunneling. The basic idea of the tunneling mechanism is as follows: We imagine that a particle-antiparticle pair is formed close to the horizon. The ingoing mode is trapped inside the horizon while the outgoing mode can quantum mechanically tunnel through the horizon and it is observed at infinity

as the Hawking flux. A method of deriving Hawking radiation based on the tunneling mechanism was suggested by Parikh and Wilczek. In this derivation, the discussion of the spectrum was not transparent. Recently, by using the tunneling mechanism, a method of deriving the black body spectrum directly was suggested by Banerjee and Majhi. But, their derivation is valid only for black holes with spherically symmetric geometry such as Schwarzschild or Reissner-Nordström black holes. We extended the simple derivation of Hawking radiation by Banerjee and Majhi on the basis of the tunneling mechanism to the case of the Kerr-Newman black hole. By using the technique of the dimensional reduction near the horizon, it is shown that the 4-dimensional Kerr-Newman metric effectively becomes a 2-dimensional spherically symmetric metric near the horizon. The use of this technique in the tunneling mechanism is justified since the tunneling effect is also the quantum effect arising near the horizon region. To discuss the behavior of matter fields near the event horizon, we defined the Kruskal-like coordinates for the effective reduced metric. We showed that our final result of the black hole radiation from a rotating black hole agrees with the previous result which is based on more elaborate analyses of the tunneling mechanism.

In conclusion, we presented several arguments which clarify two of the recent derivations of Hawking radiation based on anomalies and tunneling. To be specific, we presented a simple derivation of Hawking radiation on the basis of anomalies, and we extended Banerjee and Majhi's tunneling mechanism to a Kerr-Newman black hole by using the technique of the dimensional reduction near the horizon. A unified interpretation of various derivations of the black hole radiation, which include two derivations analyzed here, remains as an interesting problem.

## Acknowledgements

I would like to thank my supervisor, Professor Kazuo Fujikawa, for teaching me the profound aspects of physics and for helpful support and encouragement throughout my graduate study.

I also thank Professor Shigefumi Naka and Professor Shinichi Deguchi for a careful reading of the manuscript and continuous encouragement. I am very grateful to Professor Katsuyoshi Tsai, Professor Takashi Mishima and Doctor Takeshi Nihei for useful comments and encouragement. I am also grateful to all the other members of elementary particles theory group at Nihon University for many discussions and encouragement.

I am thankful to Professor Satoshi Iso and Doctor Hiroshi Umetsu for valuable discussions and helpful comments. I am happy to thank Professor Rabin Banerjee, Doctor Shailesh Kulkarni, Doctor Bibhas Ranjan Majhi, Doctor Sujoy Kumar Modak, Doctor Debraj Roy and other members of S. N. Bose National Centre for Basic Sciences for their hospitality during my stay there and for many useful discussions. I would like to appreciate my family and my friends for their encouragement and helpful support.

Finally, I gratefully acknowledge the financial support by the Strategic Project for Academic Research of Nihon University.

# Appendix

## A Killing Vectors and Null Hypersurfaces

In this appendix, we review basic properties of both Killing vectors and null hypersurfaces. To begin with, we discuss Killing vectors. On a 3-dimensional spacelike hypersurface $\Sigma$, the total energy-momentum vector is given by

$$P^{\mu} = \int_{\Sigma} T^{\mu\nu} d\Sigma_{\nu}, \tag{A.1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. This definition loses a physical meaning in the curved space. In general, the global energy or momentum conservation laws cannot be maintained. However, when there exist particular vectors, the corresponding conservation laws can be maintained.

To find these laws, we consider a quantity given by

$$P_{\xi}(\Sigma) = \int_{\Sigma} \xi_{\mu} T^{\mu\nu} d\Sigma_{\nu}, \tag{A.2}$$

where $\xi^{\mu}$ is an arbitrary vector. We note that this is a scalar quantity. Now, we consider the volume $V$ enclosed by two surfaces $\Sigma$ and $\Sigma'$ (see Fig. 3.2 in Section 3.1). According to the Gauss theorem, we obtain

$$\begin{aligned} P_{\xi}(\Sigma') - P_{\xi}(\Sigma) &= \int_{V} \nabla_{\nu}(\xi_{\mu} T^{\mu\nu}) dV & \text{(A.3)} \\ &= \int_{V} \left[(\nabla_{\nu}\xi_{\mu}) T^{\mu\nu} + \xi_{\mu}(\nabla_{\nu} T^{\mu\nu})\right] dV & \text{(A.4)} \\ &= \frac{1}{2}\int_{V} (\nabla_{\nu}\xi_{\mu} + \nabla_{\mu}\xi_{\nu}) T^{\mu\nu} dV, & \text{(A.5)} \end{aligned}$$

where we used both the local conservation law of the energy-momentum tensor

$$\nabla_{\nu} T^{\mu\nu} = 0, \tag{A.6}$$

and the fact that $T^{\mu\nu}$ is a symmetric tensor, i.e.,

$$(\nabla_{\nu}\xi_{\mu}) T^{\mu\nu} = \frac{1}{2}\left(\nabla_{\nu}\xi_{\mu} + \nabla_{\mu}\xi_{\nu}\right) T^{\mu\nu}. \tag{A.7}$$

If we choose the vector $\xi_{\mu}$ as that the vector satisfies

$$\nabla_{\nu}\xi_{\mu} + \nabla_{\mu}\xi_{\nu} = 0, \tag{A.8}$$

the quantity $P_\xi(\Sigma)$ is conserved, and we thus find that there is a corresponding symmetry in the system. The particular vector and the corresponding equation (A.8) are respectively called the "Killing vector" and the "Killing equation". In other words, when there are symmetries in the system, there exist the corresponding conserved quantities and the corresponding Killing vectors. The conserved quantities are identified at asymptotic infinity.

Next we would like to discuss hypersurfaces. To begin with, we define $\mathscr{S}(x)$ as a smooth function of the space-time coordinates $x^\mu$ and consider a family of hyperspaces

$$\mathscr{S} = \text{constant}. \tag{A.9}$$

The vector normal to the hypersurface is given by

$$l = F(x)\,(g^{\mu\nu}\partial_\nu \mathscr{S})\,\frac{\partial}{\partial x^\mu}, \tag{A.10}$$

where $F(r)$ is an arbitrary non-zero function. If the relation

$$l^2 = 0, \tag{A.11}$$

is satisfied for a particular hypersurface $\mathscr{N}$, then $\mathscr{N}$ is called a "null hypersurface".

As an example, we consider the case of Schwarzschild background. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\varphi^2. \tag{A.12}$$

By using the advanced time

$$v = t + r_*, \tag{A.13}$$

the Schwarzschild metric (A.12) in the ingoing Eddington-Finkelstein coordinates $(v, r, \theta, \varphi)$ is rewritten as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2drdv + r^2 d\Omega^2. \tag{A.14}$$

Then a surface is defined by

$$\mathscr{S} = r - 2M. \tag{A.15}$$

By substituting (A.15) into (A.10), we obtain

$$l = F(r)\left[\left(1 - \frac{2M}{r}\right)\frac{\partial}{\partial r} + \frac{\partial}{\partial v}\right], \tag{A.16}$$

and

$$l^2 = g^{\mu\nu}\partial_\mu \mathscr{S} \partial_\nu \mathscr{S} F^2(r) \tag{A.17}$$

$$= \left(1 - \frac{2M}{r}\right) F^2(r). \tag{A.18}$$

Thus we find that $r = 2M$ is a null hypersurface and the relation (A.16) then becomes

$$l|_{r=2M} = F(r)\frac{\partial}{\partial v}. \tag{A.19}$$

Thus we find that interesting properties of null hypersurfaces as follows: We define $\mathscr{N}$ as a null hypersurface with a normal vector $l$. A vector $t$ which is tangent to $\mathscr{N}$ satisfies $l \cdot t = 0$. However, since $\mathscr{N}$ is null, the relation $l \cdot l = 0$ is satisfied. Therefore, the vector $l$ is itself a tangent vector, i.e., we have

$$l^\mu = \frac{dx^\mu}{d\lambda}, \tag{A.20}$$

where $x^\mu(\lambda)$ is geodesic.

According to the definition of a Killing horizon, it is known that a Killing horizon is a null hypersurface $\mathscr{N}$ with a Killing vector $\xi$ which is normal to $\mathscr{N}$.

# B The First Integral by Carter

In this appendix, we show that the first integral is represented by

$$E^2[r^4 + a^2(r^2 + 2Mr - Q^2)] - 2E(2Mr - Q^2)ap_\varphi - (r^2 - 2Mr + Q^2)p_\varphi^2 - (\mu^2 r^2 + q)\Delta = (p_r\Delta)^2 \tag{B.1}$$

for the Kerr-Newman metric given by

$$\begin{aligned} ds^2 =& -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a\sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta)dtd\varphi \\ & -\frac{a^2\Delta\sin^2\theta - (r^2+a^2)^2}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{B.2}$$

where notations are the same as in Section 2.7. This derivation was first shown by Carter [45]. In this derivation, he adopted the metric given by

$$\begin{aligned} ds^2 =& \Sigma d\theta^2 - 2a\sin^2\theta dr d\tilde{\varphi} + 2drdu + \frac{1}{\Sigma}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta d\tilde{\varphi}^2 \\ & - \frac{2a}{\Sigma}(2Mr - Q^2)\sin^2\theta d\tilde{\varphi}du - \left[1 - \frac{2Mr - Q^2}{\Sigma}\right]du^2, \end{aligned} \tag{B.3}$$

where $u$ is the retarded time. This metric agrees with the Kerr-Newman metric (B.2) under the transformations given by

$$\begin{cases} du &= dt + \dfrac{r^2+a^2}{\Delta}dr \\ d\tilde{\varphi} &= d\varphi + \dfrac{a}{\Delta}dr. \end{cases} \tag{B.4}$$

Carter considered the behavior of a particle with a mass $\mu$ and an electrical charge $e$ in the background with the metric (B.3). In this case, the general form of the Hamilton-Jacobi equation is represented as

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{ij} \left[ \frac{\partial S}{\partial x^i} - eA_i \right] \left[ \frac{\partial S}{\partial x^j} - eA_j \right], \tag{B.5}$$

where $\lambda$ is an affine parameter associated with the proper time $\tau$ and defined by

$$\tau \equiv \mu\lambda, \tag{B.6}$$

$A_\mu$ stands for the gauge field (the electrical potential) and $S$ is the Jacobi action. If there is a separable solution, then in terms of the already known constants of the motion it must take the form

$$S = -\frac{1}{2}\mu^2\lambda - Eu + L\tilde{\varphi} + S_\theta + S_r, \tag{B.7}$$

where $E$ and $L$ are given by

$$p_u = -E, \quad p_{\tilde{\varphi}} = L, \tag{B.8}$$

where $p_\mu$ stands for the momentum component in the direction of each coordinate variable. Here $S_\theta$ and $S_r$ are respectively functions of $\theta$ and $r$ only. In this case, it can be shown that the first integral is given by

$$p_\theta^2 + \left( aE\sin\theta - \frac{L}{\sin\theta} \right)^2 + a^2\mu^2\cos^2\theta = \mathcal{K}, \tag{B.9}$$

$$\Delta p_r^2 - 2\left[(r^2+a^2)E - aL + eQr\right] p_r + \mu^2 r^2 = -\mathcal{K}, \tag{B.10}$$

where $\mathcal{K}$ is a constant.

Here we would like to know how the relations (B.9) and (B.10) are changed when we use the Kerr-Newman metric (B.2) instead of the metric (B.3). Since we consider an electrically neutral particle, we have $e = 0$. By differentiating (B.7), we obtain

$$dS = -\frac{1}{2}\mu^2 d\lambda - Edu + Ld\tilde{\varphi} + \left( \frac{\partial S_\theta}{\partial \theta} \right) d\theta + \left( \frac{\partial S_r}{\partial r} \right) dr. \tag{B.11}$$

Now we recall the transformations (B.4) which is used in order to derive the Kerr-Newman metric from (B.3). By substituting (B.4) into (B.11), we obtain

$$\begin{aligned} dS &= -\frac{1}{2}\mu^2 d\lambda - E\left(dt + \frac{r^2+a^2}{\Delta}dr\right) + L\left(d\varphi + \frac{a}{\Delta}dr\right) + \left(\frac{\partial S_\theta}{\partial\theta}\right)d\theta + \left(\frac{\partial S_r}{\partial r}\right)dr \\ &= -\frac{1}{2}\mu^2 d\lambda - Edt + Ld\varphi + \left(\frac{\partial S_\theta}{\partial\theta}\right)d\theta + \left(-\frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}\Phi + \frac{\partial S_r}{\partial r}\right)dr. \end{aligned} \tag{B.12}$$

Then the momenta conjugate to $\theta$ and $r$ are given by

$$p_\theta = \frac{\partial S_\theta}{\partial\theta}, \qquad p_r = \frac{\partial S_r}{\partial r}. \tag{B.13}$$

By using these relations, the relation (B.12) becomes

$$dS = -\frac{1}{2}\mu^2 d\lambda - Edt + Ld\varphi + p_\theta d\theta + \left(-\frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}L + p_r\right)dr. \tag{B.14}$$

By comparing between (B.11) and (B.14), we find that a new radial momentum $p'_r$ is given by

$$p'_r = p_r - \frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}L, \tag{B.15}$$

under the transformations (B.4). The relation (B.15) is also written as

$$p_r = p'_r + \frac{1}{\Delta}\left[(r^2+a^2)E - aL\right]. \tag{B.16}$$

If the constant $\mathcal{K}$ in the relation (B.9) is rewritten as

$$\mathcal{K} = q + (L - aE)^2, \tag{B.17}$$

we obtain

$$p_\theta^2 + \left(aE\sin\theta - \frac{p_\varphi}{\sin\theta}\right)^2 + a^2\mu^2\cos^2\theta = q + (p_\varphi - aE)^2 \tag{B.18}$$

$$\Leftrightarrow \qquad q = \cos^2\theta\left[a^2(\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2\theta}\right] + p_\theta^2. \tag{B.19}$$

Thus Carter's kinetic constant is used in (2.6.26) in Section 2.6. By substituting (B.17) into (B.10), we obtain

$$\Delta p_r^2 - 2[(r^2+a^2)E - aL]p_r + \mu^2 r^2 = -q - p_\varphi^2 + 2aEp_\varphi - a^2E^2. \tag{B.20}$$

By using the transformation (B.16) in (B.20), we can obtain

$$\begin{aligned} &\Delta\left(p'_r + \frac{1}{\Delta}\left[(r^2+a^2)E - ap_\varphi\right]\right)^2 - 2[(r^2+a^2)E - ap_\varphi]\left(p'_r + \frac{1}{\Delta}\left[(r^2+a^2)E - ap_\varphi\right]\right) \\ &+ \mu^2r^2 + q + p_\varphi^2 - 2aEp_\varphi + a^2E^2 = 0 \end{aligned} \tag{B.21}$$

$$\Leftrightarrow \quad E^2[r^4 + a^2(r^2 + 2Mr - Q^2)] - 2E(2Mr - Q^2)ap_\varphi - (r^2 - 2Mr + Q^2)p_\varphi^2 - (\mu^2r^2 + q)\Delta = (\Delta p'_r)^2 \tag{B.22}$$

# C Bogoliubov Transformations

In this appendix, we show that the Bogoliubov transformations (3.1.30) and (3.1.31),

$$\boldsymbol{b}_i = \sum_j (\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger), \tag{C.1}$$

$$\boldsymbol{b}_i^\dagger = \sum_j (\alpha_{ij} \boldsymbol{a}_j^\dagger - \beta_{ij} \boldsymbol{a}_j), \tag{C.2}$$

are the inverse transforms of (3.1.28) and (3.1.29) given by

$$\boldsymbol{a}_i = \sum_j (\boldsymbol{b}_j \alpha_{ij} + \boldsymbol{b}_j^\dagger \beta_{ij}^*), \tag{C.3}$$

$$\boldsymbol{a}_i^\dagger = \sum_j (\boldsymbol{b}_j \beta_{ij} + \boldsymbol{b}_j^\dagger \alpha_{ij}^*). \tag{C.4}$$

By substituting both (C.3) and (C.4) into the right-hand side of (C.1), we obtain

$$\sum_j (\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger) \tag{C.5}$$

$$= \sum_{j,k} \left[ \alpha_{ij}^* \left( \alpha_{jk} \boldsymbol{b}_k + \beta_{ik}^* \boldsymbol{b}_k^\dagger \right) - \beta_{ij}^* \left( \beta_{jk} \boldsymbol{b}_k + \alpha_{ik}^* \boldsymbol{b}_k^\dagger \right) \right] \tag{C.6}$$

$$= \sum_k \left[ \sum_j \left( \alpha_{ij}^* \alpha_{jk} - \beta_{ij}^* \beta_{jk} \right) \boldsymbol{b}_k + \sum_j \left( \alpha_{ij}^* \beta_{jk}^* - \beta_{ij}^* \alpha_{jk}^* \right) \boldsymbol{b}_k^\dagger \right]. \tag{C.7}$$

Now we recall the orthonormal condition (3.1.14) for $\{f_i\}$ and $\{f_i^*\}$, i.e.,

$$\rho(f_i, f_j^*) = \frac{1}{2} i \int_\Sigma (f_i \nabla_\mu f_j^* - f_j^* \nabla_\mu f_i) d\Sigma^\mu = \delta_{ij}, \tag{C.8}$$

which implies the following relations

$$\rho(f_i^*, f_j) = -\delta_{ij}, \tag{C.9}$$

$$\rho(f_i, f_j) = \rho(f_i^*, f_j^*) = 0. \tag{C.10}$$

The orthonormal condition (3.1.25) for $\{p_i\}$ and $\{p_i^*\}$ is also satisfied

$$\rho(p_i, p_j^*) = \delta_{ij}. \tag{C.11}$$

By substituting the relations between $\{p_i\}$ and $\{f_i\}$,

$$p_i = \sum_k (\alpha_{ik} f_k + \beta_{ik} f_k^*), \tag{C.12}$$

$$p_j^* = \sum_l (\alpha_{jl}^* f_l^* + \beta_{jl}^* f_l), \tag{C.13}$$

into (C.11), we obtain

$$\rho(p_i, p_j^*) = \rho\left(\sum_k (\alpha_{ik} f_k + \beta_{ik} f_k^*),\ \sum_l (\alpha_{jl}^* f_l^* + \beta_{jl}^* f_l)\right) \tag{C.14}$$

$$= \sum_{k,l} \left[\alpha_{ik}\alpha_{jl}^* \rho(f_k, f_l^*) + \alpha_{ik}\beta_{jl}^* \rho(f_k, f_l) + \beta_{ik}\alpha_{jl}^* \rho(f_k^*, f_l^*) + \beta_{ik}\beta_{jl}^* \rho(f_k^*, f_l)\right], \tag{C.15}$$

By using (C.8), (C.9) and (C.10) in (C.15), we obtain

$$\rho(p_i, p_j^*) = \sum_{k,l} \left[\alpha_{ik}\alpha_{jl}^* \delta_{kl} + \beta_{ik}\beta_{jl}^*(-\delta_{kl})\right] \tag{C.16}$$

$$= \sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*). \tag{C.17}$$

From (C.11), we thus obtain

$$\sum_k \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*\right) = \delta_{ij}. \tag{C.18}$$

Similarly, when we consider the case of $\rho(p_i^*, p_j^*) = 0$, we obtain

$$\sum_k \left(\beta_{ik}^*\alpha_{jk}^* - \alpha_{ik}^*\beta_{jk}^*\right) = 0. \tag{C.19}$$

By substituting (C.18) and (C.19) into (C.7), we obtain

$$\sum_k \left[\sum_j \left(\alpha_{ij}^*\alpha_{jk} - \beta_{ij}^*\beta_{jk}\right) \boldsymbol{b}_k + \sum_j \left(\alpha_{ij}^*\beta_{jk}^* - \beta_{ij}^*\alpha_{jk}^*\right) \boldsymbol{b}_k^\dagger\right] = \sum_k \delta_{ik}\boldsymbol{b}_k \tag{C.20}$$

$$= \boldsymbol{b}_i \tag{C.21}$$

Thus we can reproduce $\boldsymbol{b}_i$ from the right-hand side of (C.1). We can similarly calculate as for $\boldsymbol{b}_i^\dagger$. Therefore we can confirm that the relations (C.1) and (C.2) are the inverse transforms of (C.3) and (C.4).

# D Solutions of Klein-Gordon Equation in Schwarzschild Spacetime

In this appendix, we show that the solutions of the Klein-Gordon equation, in (3.1.10)

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0, \tag{D.1}$$

are given by (3.1.39) and (3.1.40)

$$f_{\omega' lm} = \frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}} e^{i\omega' v} Y_{lm}(\theta, \varphi), \tag{D.2}$$

$$p_{\omega lm} = \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega}} e^{i\omega u} Y_{lm}(\theta, \varphi). \tag{D.3}$$

By using a property of the covariant derivative for a scalar field $\phi$,

$$\nabla_\nu \phi = \partial_\nu \phi, \tag{D.4}$$

the equation (D.1) becomes

$$\nabla_\mu \left(g^{\mu\nu} \partial_\nu \phi\right) = 0, \tag{D.5}$$

where $\partial_\nu$ and $\nabla_\mu$ respectively stand for an ordinary derivative and a covariant derivative. The covariant derivative for a vector field $A^\mu$ is also given by

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\nu\mu} A^\nu, \tag{D.6}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol and $\Gamma^\mu_{\nu\mu}$ is written as

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu}) \tag{D.7}$$

$$= \frac{1}{2} (g^{\mu\rho} \partial_\mu g_{\rho\nu} + g^{\mu\rho} \partial_\nu g_{\rho\mu} - g^{\mu\rho} \partial_\rho g_{\nu\mu}). \tag{D.8}$$

Since the third term is canceled by the first term by exchanging $\mu$ and $\rho$ in the third term of (D.8), the relation (D.8) becomes

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\rho} \partial_\nu g_{\rho\mu}. \tag{D.9}$$

Now we would like to show that $\partial_\nu g_{\rho\mu}$ is given by

$$\partial_\nu g_{\rho\mu} = \left(g g^{\rho\mu}\right)^{-1} \partial_\nu g, \tag{D.10}$$

where $g$ is defined by

$$g \equiv \det\left(g_{\mu\rho}\right). \tag{D.11}$$

The definition (D.11) can also be written as

$$g = \exp\left(\ln \det g_{\mu\rho}\right) \tag{D.12}$$

$$= \exp\left(\mathrm{Tr} \ln g_{\mu\rho}\right). \tag{D.13}$$

Then, the small variation of $g$ is given by

$$\delta g = \exp\left\{\mathrm{Tr}\left[\ln(g_{\mu\rho} + \delta g_{\mu\rho})\right]\right\} - \exp(\mathrm{Tr}\ln g_{\mu\rho}). \tag{D.14}$$

By using the Taylor expansion for a matrix $X$

$$\ln(X + \delta X) \approx \ln X + X^{-1}\delta X, \tag{D.15}$$

the relation (D.14) becomes

$$\delta g \approx \exp\left[\mathrm{Tr}\left(\ln g_{\mu\rho} + g^{\mu\nu}\delta g_{\nu\rho}\right)\right] - \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right] \tag{D.16}$$

$$= \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right]\exp\left[\mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\rho})\right] - \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right]. \tag{D.17}$$

By performing the Taylor expansion for $\exp\left[\mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\rho})\right]$ and retaining the first order of terms, we obtain

$$\delta g \approx \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right](1 + \mathrm{Tr} g^{\mu\nu}\delta g_{\nu\rho}) - \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right] \tag{D.18}$$

$$= \exp\left[\mathrm{Tr}(\ln g_{\mu\rho})\right]\mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\rho}) \tag{D.19}$$

$$= g \cdot \mathrm{Tr}(g^{\mu\nu}\delta g_{\nu\rho}) \tag{D.20}$$

$$= g \cdot g^{\mu\nu}\delta g_{\nu\mu}. \tag{D.21}$$

By replacing $\nu$ to $\rho$ in (D.21), we obtain

$$\delta g = g \cdot g^{\mu\rho}\delta g_{\rho\mu}. \tag{D.22}$$

From this, we find

$$\partial_\nu g = g \cdot g^{\mu\rho}\partial_\nu g_{\rho\mu} \tag{D.23}$$

$$\Leftrightarrow \qquad \partial_\nu g_{\rho\mu} = (g \cdot g^{\mu\rho})^{-1}\partial_\nu g. \tag{D.24}$$

Thus we establish the relation (D.10).

By substituting (D.10) into (D.9), we obtain

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2}g^{\mu\rho}\frac{1}{g \cdot g^{\mu\rho}}\partial_\nu g \tag{D.25}$$

$$= \frac{1}{2g}\partial_\nu g. \tag{D.26}$$

Since we have

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}) = \frac{1}{2g}\partial_\nu g, \tag{D.27}$$

the relation (D.26) becomes

$$\Gamma^{\mu}_{\nu\mu} = \frac{1}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g}). \tag{D.28}$$

By substituting (D.28) into (D.6), we obtain

$$\nabla_{\mu}A^{\mu} = \partial_{\mu}A^{\mu} + \frac{1}{\sqrt{-g}}\partial_{\nu}(\sqrt{-g})A^{\nu} \tag{D.29}$$

By replacing $\nu$ by $\mu$, the relation (D.29) becomes

$$\nabla_{\mu}A^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}A^{\mu}). \tag{D.30}$$

Thus the Klein-Gordon equation (D.5) is written as

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\cdot g^{\mu\nu}\partial_{\nu}\phi) = 0. \tag{D.31}$$

Since we consider physics in the Schwarzschild background, the metric is given by (2.1.7)

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{D.32}$$

From this metric, $g$ is given by

$$g = -r^4\sin^2\theta \tag{D.33}$$

By substituting (D.32) and (D.33) into (D.31), the Klein-Gordon equation becomes

$$\left[-\frac{r}{r-2M}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{1}{r^2}\left(-\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right]\phi = 0. \tag{D.34}$$

Here we define a quadratic angular momentum by

$$\hat{L}^2 \equiv -\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{D.35}$$

Then (D.34) is written as

$$\left[-\frac{r}{r-2M}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{1}{r^2}\hat{L}^2\right]\phi = 0. \tag{D.36}$$

Now we rewrite $\phi$ as

$$\phi = \left(Ae^{-i\omega t} + A^{*}e^{i\omega t}\right)R(r)\Theta(\theta,\varphi). \tag{D.37}$$

By substituting (D.37) into (D.36), we obtain

$$\left[-\frac{r}{r-2M}(i\omega)^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}-\frac{1}{r^2}\hat{L}^2\right]\left(Ae^{-i\omega t}+A^*e^{i\omega t}\right)R(r)\Theta(\theta,\varphi)=0. \tag{D.38}$$

By dividing the both sides by $\left(Ae^{-i\omega t}+A^*e^{i\omega t}\right)$ and transferring the third term to the right-hand side, we obtain

$$\left[\frac{r}{r-2M}\omega^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}\right]R(r)\Theta(\theta,\varphi)=\frac{R(r)}{r^2}\hat{L}^2\Theta(\theta,\varphi). \tag{D.39}$$

By dividing the both sides by $\frac{R(r)}{r^2}\Theta(\theta,\varphi)$, the equation (D.39) becomes

$$\frac{r^2}{R(r)}\left[\frac{r}{r-2M}\omega^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}\right]R(r)=\frac{1}{\Theta(\theta,\varphi)}\hat{L}^2\Theta(\theta,\varphi). \tag{D.40}$$

Since (D.40) is represented in a separable form, the equation is set to be a constant. By writing the constant as $l(l+1)$, we obtain

$$\hat{L}^2\Theta(\theta,\varphi)=l(l+1)\Theta(\theta,\varphi), \tag{D.41}$$

$$\left[\frac{r}{r-2M}\omega^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}-\frac{l(l+1)}{r^2}\right]R(r)=0. \tag{D.42}$$

From (D.41), we can expand $\Theta(\theta,\varphi)$ by

$$\Theta(\theta,\varphi)=\sum_m B_{lm}Y_{lm}(\theta,\varphi), \tag{D.43}$$

where $B_{lm}$ is an integration constant and $Y_{lm}(\theta,\varphi)$ stands for the spherical harmonics. In the equation (D.42), we use the definition

$$R'(r_*)\equiv rR(r), \tag{D.44}$$

where $r_*$ is the tortoise coordinate defined by

$$r_*\equiv r+2M\ln\left|\frac{r}{2M}-1\right|. \tag{D.45}$$

Then for this transformation we

$$\frac{\partial}{\partial r}=\left(\frac{r}{r-2M}\right)\frac{\partial}{\partial r_*}. \tag{D.46}$$

Under these transformations, the equation (D.42) becomes

$$\frac{1}{r}\left[\frac{r}{r-2M}\omega^2+\left(\frac{r}{r-2M}\right)\frac{\partial^2}{\partial r_*^2}-\frac{2M}{r^3}-\frac{1}{r^2}l(l+1)\right]R'(r_*)=0. \tag{D.47}$$

By dividing the both sides of (D.47) by $\frac{1}{r}\left(\frac{r}{r-2M}\right)$, we obtain

$$\frac{\partial^2}{\partial \tilde{r}^2}R'(r_*) + \left[\omega^2 - \frac{1}{r^2}\left\{\frac{2M}{r} + l(l+1)\right\}\left(1-\frac{2M}{r}\right)\right]R'(r_*) = 0. \tag{D.48}$$

As stated in Section 3.1, the solutions $f_{\omega' lm}$ and $p_{\omega lm}$ are partial waves at $r \to \infty$. Thus, by taking $r \to \infty$ in (D.48), we obtain

$$\frac{\partial}{\partial r_*^2}R'(r_*) + \omega^2 R'(r_*) = 0. \tag{D.49}$$

The solution for this equation is given by

$$R'(r_*) = C_{\omega l}e^{-i\omega r_*} + C^*_{\omega l}e^{i\omega r_*}, \tag{D.50}$$

where $C_{\omega l}$ is an integration constant. By substituting (D.44) into (D.50), we obtain

$$R(r) = \frac{1}{r}\left(C_{\omega l}e^{-i\omega r_*} + C^*_{\omega l}e^{i\omega r_*}\right). \tag{D.51}$$

By substituting both (D.43) and (D.51) into (D.37), we thus obtain

$$\phi = \sum_{\omega,l,m}\frac{1}{r}\left(AC_{\omega l}e^{-i\omega(t+r_*)} + AC^*_{\omega l}e^{-i\omega(t-r_*)} + A^*C_{\omega l}e^{i\omega(t-r_*)} + A^*C^*_{\omega l}e^{i\omega(t+r_*)}\right)B_{lm}Y_{lm}(\theta,\varphi). \tag{D.52}$$

We use affine parameters defined by

$$v \equiv t + r_*, \tag{D.53}$$

$$u \equiv t - r_*, \tag{D.54}$$

where $v$ and $u$ are respectively called the advanced time and the retarded time. By putting together the integration constants in (D.52) except for the normalization constant $\frac{1}{\sqrt{2\pi\omega}}$, which is often used in the Klein-Gordon equation, we can thus write the partial waves as

$$f_{\omega' lm} = \frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}}e^{i\omega' v}Y_{lm}(\theta,\varphi), \tag{D.55}$$

$$p_{\omega lm} = \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega'}}e^{i\omega u}Y_{lm}(\theta,\varphi), \tag{D.56}$$

where we regarded $F_{\omega'}(r)$ and $P_\omega(r)$ as not integration constants but rather "integration variables" which depend on $r$, because we want to take into account the small effect of $r$ arising from the approximation by setting $r \to \infty$.

# E Calculation of Bogoliubov Coefficients

The Bogoliubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are given by (3.1.47) and (3.1.48)

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dv e^{-i\omega' v} p_\omega, \tag{E.1}$$

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dv e^{i\omega' v} p_\omega. \tag{E.2}$$

When the partial wave $p_\omega$ is represented as (3.1.55),

$$p_\omega^{(2)} \sim \begin{cases} 0, & (v > v_0), \\ \dfrac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i\dfrac{\omega}{\kappa}\ln\left(\dfrac{v_0 - v}{CD}\right)\right], & (v \le v_0), \end{cases} \tag{E.3}$$

we show that the corresponding $\alpha_{\omega\omega'}^{(2)}$ and $\beta_{\omega\omega'}^{(2)}$ are given by (3.1.57) and (3.1.58), i.e.,

$$\alpha_{\omega\omega'}^{(2)} \approx \frac{1}{2\pi} P_\omega^- (CD)^{\frac{i\omega}{\kappa}} e^{-i\omega' v_0} \left(\sqrt{\frac{\omega'}{\omega}}\right) \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (-i\omega')^{-1+\frac{i\omega}{\kappa}}, \tag{E.4}$$

$$\beta_{\omega\omega'}^{(2)} \approx -i\alpha_{\omega(-\omega')}^{(2)}. \tag{E.5}$$

To begin with, by substituting (E.3) into (E.1)

$$\alpha_{\omega\omega'}^{(2)} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}(r)} \int_{-\infty}^{v_0} dv \frac{P_\omega(r)}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right] e^{-i\omega' v}, \tag{E.6}$$

where $F_{\omega'}(r)$ and $P_\omega(r)$ are integration variables which take into account the small effect of $r$, and we collectively rewrote them as $P_\omega(r)$. Since we now consider the region near the horizon $r = 2M$ we use $P_\omega^- \equiv P_\omega(2M)$. Thus the relation (E.6) becomes

$$\alpha_{\omega\omega'}^{(2)} = \frac{1}{2\pi} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv (v_0 - v)^{-i\frac{\omega}{\kappa}} e^{-i\omega' v}. \tag{E.7}$$

By integrating over the variable defined as $v_0 - v = x$, we obtain

$$\alpha_{\omega\omega'}^{(2)} = \frac{1}{2\pi} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0} \int_0^{\infty} dx x^{-i\frac{\omega}{\kappa}} e^{-(-i\omega')x}. \tag{E.8}$$

By using a formula of the gamma function,

$$\Gamma(\varepsilon) t^{-\varepsilon} = \int_0^{\infty} ds s^{\varepsilon - 1} e^{-ts}, \tag{E.9}$$

we finally obtain (E.4),

$$\alpha_{\omega\omega'}^{(2)} = \frac{1}{2\pi} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0} \Gamma(1 - i\frac{\omega}{\kappa})(-i\omega')^{1-i\frac{\omega}{\kappa}}. \tag{E.10}$$

Similarly, we can show the relation (E.5) for $\beta_{\omega\omega'}^{(2)}$.

# Bibliography


[1] S.W. Hawking, Nature **248**, 30 (1974).

[2] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[3] S.P. Robinson and F. Wilczek, Phys. Rev. Lett. **95**, 011303 (2005).

[4] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. Lett. **96**, 151302 (2006).

[5] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. D**74**, 044017 (2006).

[6] K. Umetsu, Prog. Theor. Phys. **119** 849 (2008).

[7] M.K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).

[8] R. Banerjee and B.R. Majhi, Phys. Lett. B**675**, 243 (2009).

[9] K. Umetsu, arXiv:0907.1420[hep-th].

[10] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber. **778** (1915).

[11] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber. **844** (1915).

[12] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech., **189** (1916).

[13] W. Israel, Phys. Rev. **164**, 1776 (1967).

[14] B. Carter, Phys. Rev. Lett. **26**, 331 (1971).

[15] S.W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, (Cambridge University Press, 1975), §9.3.

[16] R. Ruffini and J.A. Wheeler, Phys. Today, **24**, 30 (1971).

[17] R. Penrose, Riv. Nuovo Cim.1, **252**, (1969).

[18] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).

[19] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D**48**, 1506 (1993).

[20] R.C. Myers and M.J. Perry, Ann. Phys. **172**, 304 (1986).

[21] R. Emparan and H.S. Reall, Phys. Rev. **D65**, 084025 (2002).

[22] R. Penrose, *Conformal treatment of infinity*, in B. S. DeWitt and C. M. DeWitt eds, *Relativity, Groups and Topology*,(Gordon and Breach, New York, 1964), 565.

[23] P.K. Townsend, arXiv: gr-qc/9707012.

[24] J.A. Wheeler, Phys. Rev. **97**, 511 (1955).

[25] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[26] A.S. Eddington, Nature **113**, 192 (1924).

[27] D. Finkelstein, Phys. Rev. **110**, 965 (1958).

[28] M.D. Kruskal, Phys. Rev. **119**, 1743 (1960).

[29] G. Szekeres, Publ. Mat. Debrecen **7**, 285 (1960).

[30] R. Penrose, Riv. Nuovo Cim. **1**, 252 (1969).

[31] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).

[32] S.W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971).

[33] W.C. Misner, Phys. Rev. Lett. **28**, 994 (1972).

[34] W.H. Press and S.A. Teukolsky, Nature **238**, 211 (1972).

[35] W. Unruh, Phys. Rev. Lett. **31**, 1265 (1973).

[36] R. Guven, Phys. Rev. D**16**, 1706 (1977).

[37] J.M. Greif, Junior thesis, Princeton University, *unpublished* (1969).

[38] J.D. Bekenstein, Phys. Rev. D**7**, 2333 (1973).

[39] J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. **31**, 161 (1973).

[40] R.M. Wald, *General Relativity*, (The University of Chicago Press, 1984), §12.5.

[41] C.E. Shannon, Bell Syst. Tech. J. **27**, 379 (1948).

[42] C.E. Shannon and W. Weaver, *The mathematical theory of communication*, (Urbana: University of Illinois Press, 1949).

[43] L. Brillouin, *Science and Information Theory*, (Acad. Press, Inc., New York, 1956), §12.1.

[44] D. Christodoulou and R. Ruffini, Phys. Rev. D**4**, 3552 (1971).

[45] B. Carter, Phys. Rev. **174**, 1559 (1968).

[46] J.D. Bekenstein, Lett. Nuovo Cim. **4**, 737 (1972).

[47] W. Greiner, L. Neise and H. Stöcker, *THERMODYNAMICS AND STATISTICAL MECHANICS*, (Springer-Verlag New York, Inc. 1995), Chapter 5.

[48] B.F. Schutz, *A first course in general relativity*, (Cambridge Univ. Press, 1985), §11.5.

[49] P.A. M. Dirac, *General theory of relativity*, (Princeton Univ. Press, 1996).

[50] B.S. DeWitt, Phys. Rept. C**19**, 295 (1975).

[51] S.W. Hawking, Mon. Not. Roy. Astron. Soc. **152**, 75 (1971).

[52] B.J. Carr and S.W. Hawking, Mon. Not. Roy. Astron. Soc. **168**, 399 (1974).

[53] J.D. Bekenstein, Phys. Rev. D**9**, 3292 (1974).

[54] S.W. Hawking, Commun. Math. Phys. **25**, 152 (1972).

[55] G.W. Gibbons and S.W. Hawking, Phys. Rev. D**15**, 2752 (1977).

[56] L. Smarr, Phys. Rev. Lett. **30**, 71 (1973); **30**, 521 (1973)(E).

[57] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **72**, 957 (1994).

[58] S.W. Hawking and G.T. Horowitz, Class. Quant. Grav. **13**, 1487 (1996).

[59] S.M. Christensen and S.A. Fulling, Phys. Rev. D**15**, 2088 (1977).

[60] R.A. Bertlmann and E. Kohlprath, Ann. Phys **288**, 137 (2001).

[61] K. Fujikawa, Phys. Rev. A**79**, 032334 (2009).

[62] K. Fujikawa, Phys. Rev. A**80**, 012315 (2009).

[63] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).

[64] S. Mukohyama, M. Seriu and H. Kodama, Phys. Rev. D**55**, 7666 (1997).

[65] H. Terashima, Phys. Rev. D**61**, 104016 (2000).

[66] A. Strominger and C. Vafa, Phys. Lett. B**379**, 99 (1996).

[67] C.G. Callan and J.M. Maldacena, Nucl.Phys. B**472**, 591 (1996).

[68] K. Murata and J. Soda, Phys. Rev. D**74**, 044018 (2006).

[69] E. Nöther, Nachr. d. Köning. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, 235 (1918). (English translation by M.A. Tavel, arXiv: physics/0503066).

[70] H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4**, 49 (1949).

[71] J.S. Bell and R. Jackiw, Nuovo Cim. A**60**, 47 (1969).

[72] S.L. Adler, Phys. Rev. **177**, 2426 (1969).

[73] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).

[74] K. Fujikawa, Phys. Rev. D**21**, 2848 (1980); D**22**, 1499 (1980)(E).

[75] K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies*, (Oxford Univ. Press, 2004).

[76] R. Banerjee and S. Kulkarni, Phys. Rev. D**77**, 024018 (2008).

[77] J. Wess and B. Zumino, Phys. Lett. B**37**, 95 (1971).

[78] R. Banerjee and S. Kulkarni, Phys. Lett. B**659**, 827 (2008).

[79] A.M. Polyakov, Phys. Lett. B**103**, 207 (1981).

[80] H. Leutwyler, Phys. Lett. B**153**, 65 (1985).

[81] K. Fujikawa, Phys. Rev. Lett. **44**, 1733 (1980).

[82] L. Alvarez-Gaume and E. Witten, Nucl. Phys. B**234**, 269 (1984).

[83] K. Fujikawa, M. Tomiya and O. Yasuda, Z. Phys. C**28**, 289 (1985).

[84] R. Bertlmann and E. Kohlprath, Ann. of Phys. **288**, 137 (2001).

[85] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976).

[86] W.A. Bardeen and B. Zumino, Nucl. Phys. B**244**, 421 (1984).

[87] S. Iso, Int. J. Mod. Phys. A**23**, 2082, (2008).

[88] A. Shirasaka and T. Hirata, arXiv:0804.1910[hep-th].

[89] L. Bonora, M. Cvitan, S. Pallua and I. Smolic, JHEP **0812**, 012 (2008).

[90] R. Banerjee and S. Kulkarni, Phys. Rev. D**79**, 084035 (2009).

[91] K. Fujikawa, Int. J. Mod. Phys. A**24**, 3306, (2009).

[92] T. Morita, Phys. Lett. B**677**, 88 (2009).

[93] L. Bonora, M. Cvitan, S. Pallua and I. Smolic, Phys. Rev. D**80**, 084034 (2009).

[94] S. Iso, T. Morita and H. Umetsu, JHEP **0704**, 068 (2007).

[95] A.P. Porfyriadis, Phys. Rev. D**79**, 084039 (2009).

[96] S. Nam and J.D. Park, Class. Quant. Grav. **26**, 145015 (2009).

[97] A.P. Porfyriadis, Phys. Lett. B**675**, 235 (2009).

[98] S.W. Wei, R. Li, Y.X. Liu and J.R. Ren, arXiv:0904.2915[hep-th].

[99] S. Iso, T. Morita and H. Umetsu, Phys. Rev. D**75**, 124004 (2007).

[100] S. Iso, T. Morita and H. Umetsu, Phys. Rev. D**76**, 064015 (2007).

[101] S. Iso, T. Morita and H. Umetsu, Nucl. Phys. B**799**, 60 (2008).

[102] S. Iso, T. Morita and H. Umetsu, Phys. Rev. D**77**, 045007 (2008).

[103] J.Y. Zhang and Z. Zhao, Mod. Phys. Lett. A**20**, 1673 (2005).

[104] J.Y. Zhang and Z. Zhao, Phys. Lett. B**618**, 14 (2005).

[105] Q.Q. Jiang, S.Q. Wu and X. Cai, Phys. Rev. D**73**, 064003 (2006).

[106] D. Chen and S. Yang, Mod. Phys. Lett. A**22**, 2611 (2007).

[107] R. Banerjee and S.K. Modak, JHEP **0905**, 063 (2009).

[108] P. Painlevé, C. R. Acad. Sci. **173**, 677 (1921).

[109] P. Kraus and F. Wilczek, Mod. Phys. Lett. A**9**, 3713 (1994).

[110] R. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. **116**, 1322 (1959).

[111] R. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. **122**, 997 (1961).

[112] R. Arnowitt, S. Deser and C.W. Misner, *Gravitation: an introduction to current research*, (Wiley, New York, 1962) Chapter 7, 227; arXiv: gr-qc/0405109.

[113] P.C.W. Davies and S.A. Fulling, Proc. Roy. Soc. Lond. A**354**, 59 (1977).

[114] S.A. Fulling, Gen. Rel. Grav. **18**, 609 (1986).

[115] S.R. Coleman, Phys. Rev. D**11**, 2088 (1975).

[116] R. Banerjee, B.R. Majhi and E.C. Vagenas, arXiv:0907.4271[hep-th].

[117] R. Banerjee and S.K. Modak, JHEP **0911**, 073 (2009).

[118] D. Roy, Phys. Lett. B**681**, 185 (2009).